CERTAIN RESULTS ON FIXED POINT THEOREMS AND VARIATIONAL INEQUALITIES

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

APPLIED MATHEMATICS

BY

HUMA SAHPER

UNDER THE SUPERVISION OF

DR. SHAMSHAD HUSAIN

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING AND TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2015

Fed in Computer

Maulana Azad Library
T-9602
Acc. No
Aligarh Muslim University, Aligarh

24 SEP 2016

T9602

Dedicated

to

My Family

Dr. Shamshad Husain
Associate Professor

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING AND TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH – 202002, INDIA
Ph. No.: +91 9411209144
E-mail: s_husain68@yahoo.com

Certificate

*This is to certify that the thesis entitled "**Certain Results on Fixed Point Theorems a Variational Inequalities**" is the research work of Mrs. Huma Sahper carried out under n supervision and guidance. She has fulfilled the prescribed conditions given in the statues a ordinances of Aligarh Muslim University, Aligarh.*

I further certify that the work of this thesis either partially or fully has not been submitt to any other University or Institution for the award of any degree.

Chairman
Department of Applied Mathematics
Z.H. College of Engg. & Tech.
A.M.U., Aligarh

(Dr. Shamshad Husai
Supervisor

Table of Contents

Acknowledgements

First and foremost, I must bow in reverence to Almighty **'ALLAH'** *the cherisher and sustainer to whom we entirely owe all our capabilities, strength, courage and knowledge without His Blessings nothing could be done.*

Now, I wish to place on record my profound gratitude and indebtedness to my reverend supervisor Dr. Shamshad Husain, Associate Professor, Department of Applied Mathematics, Aligarh Muslim University, Aligarh, whose valuable guidance and explicit enthusiasm helped me in several ways at various stages. His keen interest, sharp observations, careful and reasoned criticism remained a source of encouragement throughout the course of this study.

I wish to express my deep sense of gratitude to Prof. Mohammad Saleem (present Chairman) and Prof. Mumtaz Ahmad Khan (Ex-Chairman), Department of Applied Mathematics, Aligarh Muslim University, Aligarh, for providing me various departmental facilities and creating a congenial atmosphere to carry out this work. I also avail this opportunity to express my thanks to all my teachers for their encouragement and inspiration.

I wish to express my profound gratitude and indebtedness to my father and teacher Prof. Mohammad Imdad, Department of Mathematics, Aligarh Muslim University, Aligarh, who has been an unfailing source of inspiration and affection. His constructive criticism at the time of drafting of the thesis resulted in many improvements of both the contents and the presentation. His deep knowledge and enthusiasm for the subject was a continuous source of inspiration to me and no words can truly convey my gratitude.

I wish to express my thanks to all my senior colleagues, friends and co-workers, especially to Dr. Sunny Chauhan, Dr. Sanjeev Gupta, Dr. Mohd. Hasan, Dr. Anupam Sharma, Mr. Aftab Alam, Mr. Md. Ahmadullah and Miss Nisha Singh who spared their valuable time to go through this manuscript either partly or completely and made some worthwhile suggestions.

Mere words are insufficient to express my love and affection for my mother Mrs. Wakilun Nisa for her unflinching support and understanding. I express my rare indebt-

ness to her, whose countless blessing, deep love and affection always worked as a hidden power behind every thing I did. My special thanks also goes to my brother Mr. Azhar Jamal and my sister Miss Adeeba Fatima for all their love and encouragement.

I would like to express my special thanks to my husband Mr. Aafaq Ahmad, who shared my family responsibilities and encouraged me to concentrate on my studies. Without his help and encouragement, this work would not have been completed. I am also highly grateful to my son Mohammad Aariz Khan for being tolerant with my ignorance all the while and remaining with me as a constant source of emotional support throughout this work.

I also accord my thanks to the non-teaching staff of the Department of Applied Mathematics, Aligarh Muslim University, Aligarh, for their co-operation.

Finally, I express my indebtedness to my glorious and esteemed institution, Aligarh Muslim University, Aligarh and U.G.C., for providing me financial assistance in the form of scholarship during my research programme.

Dated: 1/10/2015

Huma Sapper
(Huma Sapper)

List of symbols

$A = B$, $A \neq B$	: Equality and Inequality for sets
$[a, b]$, $[a, b)$, etc.	: Intervals on the real line
$f(x)$ or fx	: Image of x under f
inf	: infimum (or greatest lower bound)
sup	: supremum (or least upper bound)
$\lim\limits_{n\to\infty} x_n = x$	: limit of the sequence $\{x_n\}$
max	: Maximum
min	: Minimum
$\mathbb{N}$	: The set of all natural numbers
$\mathbb{R}$	: The set of all real numbers
$\mathbb{R}^+$	: The set of all positive real numbers
$\mathbb{R}_0^+$	: The set of all non-negative real numbers
$\mathbb{R}^n$	: n-dimensional Euclidean space
$\mathbb{Q}$	: The set of all rational numbers
$\lvert x\rvert$	: Absolute value of x
$\lVert\cdot\rVert$	: Norm
For a mapping $T : X \to X$,	
$F(T)$	: The set of all fixed points of T
I	: The identity map

T^n	: ToT^{n-1}

For a metric space (X, d)

$B(a, r)$	: $\{x \in X \; : \; d(x, a) < r\}$
$\overline{B}(a, r)$	: $\{x \in X \; : \; d(x, a) \leq r\}$
$d(a, b)$	: Distance between a and b
$d(x, A)$	: Distance between the point x and set A
$\emptyset$	: Empty set

Preface

Fixed point theory is a very wide topic of mathematical research as it belongs to several mathematical domains such as: Functional Analysis, Variational Inequalities, Operator Theory, Topology, Geometry and Algebraic Topology. Fixed point theorems have extensive applications in several branches of Mathematics, Engineering and Physical Sciences. Many concrete research problems in science and engineering can be reduced to fixed point problems. Most of the theorems ensuring the existence of solutions to Differential Equations, Integral Equations or other Operator Equations can be proved using fixed point theorems. Fixed point theorems are also used in diverse areas of mathematical applications such as: Mathematical Economics, Game Theory, Fluid Flow, Approximation Theory, Random Differential Equations etc.

The theory presented here has great contributors. Those who developed the classical theory include the celebrated mathematicians, namely: L. E. J. Brouwer [28], S. Banach [21] and J. Schauder [177]. The first ever fixed point theorem for contraction mappings in metric spaces was proved by S. Banach (a Polish Mathematician) in 1922, which is now popularly referred as Banach Contraction Principle. The enormous utility of this principle is an established fact. This result is one of the most fruitful results of metric fixed point theory whose significance lies in its vast applicability in diverse disciplines. Generalization of the Banach contraction principle has been a heavily investigated subject in metric fixed point theory which continues to be an active area of research.

In recent past, variational inequality theory has rapidly developed into an elegant and fascinating branch of applicable mathematics. Variational inequalities do arise in various models for a large number of mathematical and physical problems. Till now an extensive interest has been shown in developing various extensions and generalizations of variational inequalities related to set-valued operators, nonconvex optimization and nonmonotone operators. There are three different genuine motivations to study variational inequalities. (i) Mathematical Modelling: to convert the problems of real life or the problems from science, engineering and social science into a variational inequal-problem. (ii) Existence Theory: to study the existence of solutions of variational equalities. (iii) Numerical Methods: to construct the algorithms for computing the proximate solutions of variational inequalities, which converge to the exact solution.

The main objective of this thesis is three fold:

- to prove some common fixed point theorems in Menger probabilistic metric spaces (in short, Menger PM-spaces) and Menger probabilistic quasi-metric spaces (in short, Menger PQM-spaces) (Chapter 2).
- to prove some multi-tupled coincidence and fixed point theorems for nonlinear contractions in ordered metric spaces (Chapters 3 and 4).
- to study the convergence of iterative algorithms and existence of solutions for certain classes of variational inequalities in the setting of Hilbert and Banach spaces (Chapters 5 and 6).

The thesis consists of six chapters wherein each chapter is further divided into var-us sections. The definitions, examples, remarks, lemmas, theorems, corollaries etc. ve been specified with the double decimal numbers. The first figure denotes the mber of the chapter, second represents the section in the chapter and third points t the number of the definition, the example, or the theorem as the case may be a particular chapter. For example, the number like 4.2.3 indicates Theorem (or mma/Proposition/Remark/Corollary/Definition) three appearing in section two of e chapter four. As usual the numbers in big brackets refer to the references listed the bibliography. The first section of each chapter provides an introduction to its ntents.

As usual, Chapter 1 is introductory in nature wherein we have discussed the histor-l development of fixed point theory and variational inequalities and visited relevant eliminary concepts, definitions and important results relevant to our subsequent dis-ssions.

Chapter 2 is devoted to certain results on coincidence and common fixed point theo-ms in Menger PM and PQM-spaces. Section 2.2 contains preliminaries which include me basic notions and core results. Details of implicit functions due to Singh and Jain 39] as well as Imdad and Ali [89] will be presented in Section 2.3. In Section 2.4, we ove some common fixed theorems for two pairs of converse commuting mappings in enger PM-spaces using the implicit functions discussed in Section 2.3 while Section 5 is devoted to some illustrative examples demonstrating the results proved in Section

2.4. In Section 2.6, we obtain a common fixed point theorem for a pair of finite number of self mappings in Menger PQM-space using weak compatibility.

In Chapter 3, some even tupled coincidence and fixed point theorems in ordered metric spaces have been proved. As usual, Section 3.1 is devoted to a brief introduction, while Section 3.2 contains preliminaries which include some basic notions and core results. In Section 3.3, we prove existence and uniqueness of some even tupled fixed point theorems for mappings having the mixed monotone property in ordered complete metric spaces dependent on another function. In Section 3.4, we prove results on even tupled coincidence and common fixed point for mappings $F : X^n \to X$ and $g : X \to X$ without completeness (of the metric space X) and commutativity or compatibility conditions (on the mappings F and g). In the last and final section, we furnish an illustrative example to demonstrate the main result proved in Section 3.4.

Chapter 4 deals with some multi-tupled coincidence point results for mappings without the mixed monotone property. Section 4.2 mainly contains relevant preliminary concepts and known results. In Section 4.3, we rectify some discrepancies observed in recent results of Doric et al. [57], which also applies to corresponding results of Chandok et al. [34], Chandok and Tas [35] etc. In Section 4.4, we establish some n-tupled coincidence point results in ordered metric spaces for a pair of mappings without mixed monotone property satisfying contractive condition of rational type. Also, we present a result on the existence and uniqueness of common n-tupled fixed points. In the last and final section, we furnish an illustrative example to demonstrate our main result.

In Chapter 5, using the resolvent operator associated with $H(.,.)$-cocoercive operator due to Ahmad et al. [7], we introduce and study a system of generalized nonlinear quasi-variational-like inclusions and corresponding system of generalized resolvent equations in real Hilbert spaces. In Section 5.2, we include a brief introduction of $H(.,.)$-cocoercive operator and its properties investigated by Ahmad et al. [7]. In Section 5.3, we consider a system of generalized nonlinear quasi-variational-like inclusions in real Hilbert spaces. By using the resolvent operators associated with $H(,.,)$-cocoercive operators, we prove that the approximate solutions obtained by the iterative algorithms converge to the exact solutions of our system of generalized nonlinear quasi-variational-like inclusions. In Section 5.4, we establish equivalence between system of generalized nonlinear quasi-variational-like inclusions considered in Section 5.3 and system of generalized resolvent equations. This equivalence is used to suggest an iterative algorithm for finding the approximate solution of system of generalized resolvent equations. We also study the convergence of iterative sequences generated by the proposed algorithm.

In the last and final chapter, we investigate a notion of generalized accretive mapng known as generalized $H(.,.,.)$-η-cocoercive operator in q-uniformaly smooth Banach aces. In Section 6.2, we prove that the resolvent operator associated with $H(.,.,.)$-cocoercive operator is single-valued and Lipschitz continuous. Some examples are nstructed to illustrate the definition of $H(.,.,.)$-η-cocoercive operator. In Section 6.3, e consider a system of generalized set-valued mixed-quasi variational like inclusion oblem involving generalized $H(.,.,.)$-η-cocoercive operators in q-uniformaly smooth anach spaces. By using the resolvent operator technique, we construct an iterative alrithm for finding the approximate solutions of system of generalized set-valued mixedasi variational like inclusion and discuss its convergence analysis.

In the end, we give a comprehensive list of references of books, monographs, lited volumes and research papers used in carrying out this research work.

The contents of this thesis have been published/accepted/communicated in the ternational journals listed below:

(i) Generalized n-tupled fixed point theorems in partially ordered metric spaces involving an ICS map, Advances in Fixed Point Theory, 3(3), 476-492, (2013).

(ii) Common fixed point theorems for conversely commuting mappings using implicit relations, Jour. of Operators, 2013:391474, 5pp., (2013).

iii) A common fixed point theorem for weakly compatible mappings in Menger probabilistic quasi metric space, Jour. Nonlinear Anal. Appl., 2014:jnaa-00142, 9pp., (2014).

iv) Algorithm for solving a new system of generalized nonlinear quasi-variational-like inclusions in Hilbert spaces, Chinese Jour. Math., 2014:957482, 7pp., (2014).

(v) Generalized n-tupled common fixed point theorems for contractive rational type condition, British Jour. Math. Comput. Sci., 4(5), 735-748, (2014).

vi) System of Generalized Resolvent Equations involving $H(.,.)$-cocoercive operator, Internat. Math. Forum, accepted.

vii) Generalized $H(.,.,.)$-η-cocoercive operator with an application, Filomat, communicated.

iii) Comments on some recent multi-tupled fixed point results for mappings without mixed monotone property, Miskolic Math. Notes, communicated.

(ix) n-tupled coincidence and common fixed point results in ordered metric spaces without completeness and compatibility, Fixed Point Theory Appl., communicated.

The work of this thesis has also been presented in the following national/international conferences:

(i) International Conference on Recent Advances in Mathematical Biology, Analysis and Applications, held at Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India, during June 4-6, 2015.

(ii) International Conference on Advances in Pure and Applied Mathematics, held at Department of Mathematics, Jawaharlal Nehru Government College, Haripur (Manali), India, during Mar. 7-9, 2014.

(iii) 17th Annual Conference of Vijnana Parishad of India and National Seminar on Mathematics Through the Ages, held at Department of Mathematics, Govt. Digvijay P. G. College, Rajnandgaon, India, during Feb. 20-21, 2014.

(iv) National Seminar on Advances in Nonlinear Analysis and Optimization, held at School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, India, during Feb. 15-17, 2014.

(v) International Conference on Recent Trends in Algebra Analysis and its Applications, held at Department of Mathematics, Aligarh Muslim University, Aligarh, India, during Feb. 12-14, 2014.

Chapter 1

Preliminaries

1.1 Literature of Fixed Point Theorems

Fixed point theory is a natural mixture of Analysis, Topology and Geometry. In the course of last hundred years of its existence, fixed point theorem is a fully developed but still a young area of research. As mentioned in the preface fixed point theory is not only a branch of pure mathematics but it has fascinating applications in diverse areas within and beyond mathematics such as Biology, Chemistry, Economics, Engineering, Game Theory and Physics. For details one can see Collatz [48], Cronion [52], Cesari [33], Kreyszig [117], Leggett and william [123], Martin [127], Moore [134], Smart [190] etc.

Definition 1.1.1. Let X be a nonempty set and $f : X \to X$ a mapping. An element $x \in X$ is said to be a fixed point (or invariant point) of the mapping f if $fx = x$.

By a fixed point theorem, we shall understand a statement which asserts that under certain conditions (on the mapping f and on the space X) a mapping f of X admits one or more fixed points.

Perhaps Brouwer [28] was the first mathematician to formulate a fixed point theorem which states that a continuous self-mapping of a closed unit ball in n-dimensional Euclidean space has at least one fixed point. Subsequently Schauder [177] extended Brouwer's theorem to the compact convex subsets of a normed linear space. This theorem was further extended to locally convex topological vector space by Tychonoff [195]. Later on, Banach [21] obtained the fixed point theorem for contraction mappings which is very famous because its proof is simple and does not require much topological background.

efinition 1.1.2. Let (X, d) be a metric space. A self mapping f on X is called pschitzian with Lipschitz constant k if there exists a non-negative real number k such at for all $x, y \in X$,

$$d(f(x), f(y)) \leq kd(x, y).$$

) f is contraction if $0 \leq k < 1$.
) f is nonexpansive if $k = 1$.
i) f is contractive if $d(fx, fy) < d(x, y)$, for all $x, y \in X$ with $x \neq y$.
) f is isometry if $d(fx, fy) = d(x, y)$, for all $x, y \in X$.

heorem 1.1.1. [21] Let (X, d) be a complete metric space and f a contraction of X to itself. Then f has a unique fixed point, say x in X.

Many generalizations of the contractive mapping theorem of Banach were obtained Bryant [29], Rakotch [165], Kannan [97], Chatterjee [38], Ćirić [49] and many others.

The study of common fixed point of some contractive type mappings forms a subject vigorous research interest wherein a number of interesting results have been proved various authors.

efinition 1.1.3. Let X be an arbitrary set and $\mathbb{T}$ a family of mappings $T : X \to X$. point $x \in X$ is called a common fixed point for the family if $T(x) = x$ for all $T \in \mathbb{T}$.

Jungck [94] was perhaps the first mathematician who generalized Banach's fixed oint theorem by proving a common fixed point theorem, which runs as follows:

heorem 1.1.2. Let T be a continuous mapping of a complete metric space (X, d) into self. Then T has a fixed point in X if there exists $k \in (0, 1)$ and a mapping $S : X \to X$ hich commutes with T and satisfies $S(X) \subset T(X)$ and $d(Sx, Sy) \leq kd(Tx, Ty)$, for l $x, y \in X$.

Sessa [182] generalized the result of Jungck [94] and introduced the notion of weak mmutativity which is defined as follows:

efinition 1.1.4. Two self maps A and S of a metric space (X, d) are said to be weakly mmuting if for all $x \in X$,

$$d(ASx, SAx) \leq d(Ax, Sx).$$

Notice that, a commuting pair is always weakly commuting but the converse is not nerally true.

Example 1.1.1. Consider the set $X = [0, 1]$ with the usual metric. Let $Ix = x/2$ and $Sx = x/(2+x)$ for every $x \in X$. Then for all $x \in X$, $ISx = \frac{x}{4+2x}$, $SIx = \frac{x}{4+x}$; obviously S and I are not commuting. Further

$$\begin{aligned} d(ISx, SIx) &= \left|\tfrac{x}{4+2x} - \tfrac{x}{4+x}\right| = \tfrac{x^2}{(4+x)(4+2x)} \\ &\leq \tfrac{x^2}{4+2x} = \tfrac{x}{2} - \tfrac{x}{2+x} = d(Sx, Ix). \end{aligned}$$

So, S and I are weakly commuting.

Further, Jungck [95] introduced more generalized commutativity condition under the name of compatibility.

Definition 1.1.5. Two self maps A and S of a metric space (X, d) are called compatible if,

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Note that, weakly commuting mappings are compatible, but the converse need not be true as shown in the following example:

Example 1.1.2. Consider the set $X = \mathbb{R}$ with the usual metric. Let $Ix = x^3$ and $Sx = 2 - x$ for every $x \in X$. Then $|I(x_n) - S(x_n)| = |x_n - 1||x_n^2 + x_n + 2| \to 0$ iff $x_n \to 1$ and $|IS(x_n) - SI(x_n)| = 6|x_n - 1|^2 \to 0$ if $x_n \to 1$. Thus S and I are compatible but are not weakly commuting pair.

Jungck and Rhoades [96] introduced the notion of weakly compatible maps as follows:

Definition 1.1.6. Two self maps A and S of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, that is, if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

It is obvious that compatible maps are weakly compatible but the converse is not true in general.

1.2 Probabilistic Metric Spaces

The concept of Menger probabilistic metric space (in short, Menger PM-space) was initiated by Menger [128] in 1942. The idea of Menger was to use a distribution function instead of non-negative real number for the value of a metric. The notion of a probabilistic metric space corresponds to the situation when we do not know exactly the distance between two points. Thus, one thinks of the distance between two points x and y as being probabilistic with $F_{x,y}(t)$ representing the probability that the distance between x and y is less than t.

Definition 1.2.1. A mapping $F : \mathbb{R} \longrightarrow \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf \{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\mathcal{T}$ the set of all distribution functions defined on $(-\infty, \infty)$ and $\varepsilon_0(t)$ denote the specific distribution function defined by

$$\varepsilon_0(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \mathcal{T}$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 1.2.2. A Probabilistic metric space is an ordered pair $(X, \mathcal{F})$, where X is a non empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(i) (PM-1): $F_{x,y}(t) = 1$ for all $t > 0 \Longleftrightarrow x = y$;

(ii) (PM-2): $F_{x,y}(0) = 0$;

(iii) (PM-3): $F_{x,y}(t) = F_{y,x}(t)$;

(iv) (PM-4): If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$.

Remark 1.2.1. In view of the condition (PM-2) which obviously implies that $F_{x,y}(t) = 0$ for $t \leq 0$, condition (PM-1) is equivalent to statement $x = y \Longleftrightarrow F_{x,y} = \varepsilon_0$.

Remark 1.2.2. Every metric space (X, d) can always be realized as a PM-Space (of a special kind) if we set $F_{x,y}(t) = \varepsilon_0(t - d(x, y))$, for all $x, y \in X$ and $t \in \mathbb{R}$.

The condition (PM-4) of above definition is always satisfied in a metric space wherein it reduces to the ordinary triangular inequality. However, in those PM-Spaces in which the equality $F_{x,y}(t) = 1$ does not hold for any finite t, the condition (PM-4) will be satisfied vacuously.

Definition 1.2.3. A triangular inequality is said to hold universally in a PM-space iff it holds for all triples of points (distinct or not) in that space.

Let $\Delta : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a 2-place function satisfying the following conditions:

(i) (Δ-1): $0 \leq \Delta(a, b) \leq 1$,

(ii) (Δ-2): $\Delta(c, d) \geq \Delta(a, b)$, for $c \geq a, d \geq b$,

(iii) (Δ-3): $\Delta(a, b) = \Delta(b, a)$,

(iv) (Δ-4): $\Delta(1, 1) = 1$,

(v) (Δ-5): $\Delta(a, 1) > 0$, for all $a > 0$.

Remark 1.2.3. K. Menger introduced the generalized triangular inequality (also referred as Menger triangular inequality) as (PM-5): $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(t))$, where Δ is a 2-place function satisfying conditions (Δ-1) to (Δ-5). From condition (Δ-4), we see that (PM-5) contains the condition (PM-4) as a special case. There are numerous possible choices for Δ. The following six are most natural and simplest examples of 2-place functions:

(i) $\Delta_1(a, b) = \max(a + b - 1, 0)$,

(ii) $\Delta_2(a, b) = a.b$,

(iii) $\Delta_3(a, b) = \min(a, b)$,

(iv) $\Delta_4(a, b) = \max(a, b)$,

(v) $\Delta_5(a, b) = a + b - ab$,

(vi) $\Delta_6(a, b) = \min(a + b, 1)$.

Lemma 1.2.1. If a PM-space contains two distinct points, then the condition (PM-5) cannot hold universally in the space under the choice $\Delta_4 = \max$.

Lemma 1.2.2. If a PM-space is not a metric space and the condition (PM-5) holds universally in the space for some choice of Δ satisfying the conditions (Δ-1) to (Δ-5), then the function Δ has the property that there exists a number a, $0 < a < 1$, such that $\Delta(a, 1) \leq a$.

Theorem 1.2.1. If the condition (PM-5) holds universally in a PM-Space and Δ is continuous, then for any $x > 0$, $\Delta(F_{p,q}(x), 1) \leq F_{p,q}(x)$.

In view of the preceding lemmas and the fact that the three weaker functions in our list of Δ'^s satisfy $\Delta(a,1) = a$, we are led to replace the conditions (Δ-1), (Δ-4) and (Δ-5) by the following conditions:

(i) (Δ-6) : $\Delta\ (a,1) = a$ and $\Delta\ (0,0) = 0$.

Thus far we also add the associativity condition:

(ii) (Δ-7) : $\Delta\ (\Delta\ (a,b)\ ,c) = \Delta\ (a, \Delta\ (b,c))$,

which permits the extension of condition (Δ-6).

Definition 1.2.4. A Menger PM-space is a PM-space in which the condition (PM-5) holds universally for some choice of Δ satisfying (Δ-2), (Δ-3), (Δ-6) and (Δ-7).

Definition 1.2.5. A triangular norm (or a $t-$ norm) is a 2- place function $\Delta : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the conditions (Δ-2), (Δ-3), (Δ-6) and (Δ-7).

For classical examples of continuous t-norm, one may recall t-norms T_L, T_P and T_M which are respectively defined as $T_L(a,b) = \max\{a+b-1, 0\}$, $T_P(a,b) = ab$ and $T_M(a,b) = \min\{a,b\}$.

The following lemma shows that, in determining whether or not a PM-Space is a Menger PM-Space, only triple of distinct points are to be considered.

Lemma 1.2.3. If the points x, y, z are not all distinct, then the condition (PM-5) holds for the triple x, y, z under any choice of Δ satisfying (Δ-2), (Δ-3), (Δ-6) and (Δ-7).

Definition 1.2.6. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. Then
(i) A sequence $\{x_n\}$ in X is said to be convergent to x in X, if for every $\epsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n,x}(\epsilon) > 1-\lambda$, whenever $n \geq N$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$, whenever $n, m \geq N$.
(iii) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

1.3 Multi-Tupled Fixed Points in Ordered Metric Spaces

Recently, an extension of Banach's contraction principle has been considered by many authors in different metric spaces. In 2004, Ran and Reurings [166] extended the Banach's contraction principle in ordered metric spaces for continuous monotone mappings with some applications to matrix equations wherein the involved contractive condition is required to hold merely on elements which are comparable in the underlying partial ordering. Henceforth, Nieto and Lopez [143] extended results of Ran and Reuring for non-decreasing mappings which are not necessarily continuous by assuming an additional hypothesis on ordered metric space and studied existence and uniqueness of solutions of first-order differential equations. For further details in ordered metric spaces, we refer to see [2, 5, 12, 13, 50, 75, 76, 77, 93, 139, 141, 144, 154, 162].

Now, we present some basic definitions and related results to make our presentation as self-contained as possible.

Definition 1.3.1. Let X be a non-empty set. A relation '$\preceq$' on X is said to be a partial order if the following properties are satisfied:
(i) reflexive: $x \preceq x$ for all $x \in X$;
(ii) anti-symmetric: $x \preceq y$ and $y \preceq x$ implies $x = y$;
(iii) transitive: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in X$.

A non-empty set X together with a partial order '$\preceq$' is said to be an ordered set and we denote it by $(X, \preceq)$.

Definition 1.3.2. Let $(X, \preceq)$ be an ordered set. Any two elements x and y are said to be comparable elements in X if either $x \preceq y$ or $y \preceq x$ and we denote it as $x \prec\succ y$.

Clearly, the relation '$\prec\succ$' is reflexive and symmetric but not transitive in general (see, Turinci [194]).

Definition 1.3.3. Let $(X, \preceq)$ be an ordered set and $F : X \to X$ a mapping. Then F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_2) \preceq F(x_1)$.

Definition 1.3.4. Let $(X, \preceq)$ be an ordered set and $F : X \to X$, $g : X \to X$ two mappings. Then F is said to be g-nondecreasing if for all $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ implies $F(x_1) \preceq F(x_2)$ and g-nonincreasing if for all $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ implies $F(x_2) \preceq F(x_1)$.

Definition 1.3.5. [154] A triplet $(X, d, \preceq)$ is said to be an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, if d is a complete metric on X, then we say that $(X, d, \preceq)$ is an ordered complete metric space.

The study of multi-tupled fixed points in ordered metric spaces was initiated by

Bhaskar and Lakshmikantham [30], where the authors introduced the concept of coupled fixed point and mixed monotone property and proved some theorems on the existence and uniqueness of coupled fixed points, which are also viewed as coupled formulation of certain results of Nieto and Lopez [143].

Throughout this section, let us denote by X^n the product space $X \times X \times ... \times X$ of n identical copies of X.

Definition 1.3.6. [30] Let X be a non-empty set and $F : X^2 \to X$ a mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of F if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 1.3.7. [30] Let $(X, \preceq)$ be an ordered set and $F : X^2 \to X$ a mapping. Then F is said to have mixed monotone property if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument, *i.e.*, for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

In 2009, Lakshmikantham and Ćirić [119] established coupled coincidence point theorems for mappings $F : X \times X \to X$ and $g : X \to X$ by defining the concept of mixed g-monotone property. The definitions are as follows:

Definition 1.3.8. [119] Let X be a non-empty set and $F : X^2 \to X$, $g : X \to X$ two mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if

$$F(x, y) = g(x) \text{ and } F(y, x) = g(y).$$

Definition 1.3.9. [119] Let $(X, \preceq)$ be an ordered set and $F : X^2 \to X$ and $g : X \to X$ two mappings. Then F is said to have mixed g-monotone property if F is monotone g-nondecreasing in its first argument and monotone g-nonincreasing in its second argument, *i.e.*, for any $x, y \in X$,

$$x_1, x_2 \in X,\ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X,\ g(y_1) \preceq g(y_2) \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

Note that, under the restriction $g = I$ (identity mapping), Definitions 1.3.8 and 1.3.9 reduces to Definitions 1.3.6 and 1.3.7.

Definition 1.3.10. [119] Let X be a non-empty set. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commuting if

$$g(F(x, y)) = F(g(x), g(y)), \quad \text{for all } x, y \in X.$$

The notion of commutativity of mappings F and g is generalized by Choudhury and Kundu [45] by defining the concept of compatibility of mappings F and g.

Definition 1.3.11. [45] Let X be a non-empty set. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{cases} \lim\limits_{m\to\infty} d\big(g(F(x_m, y_m)), F(gx_m, gy_m)\big) = 0 \\ \lim\limits_{m\to\infty} d\big(g(F(y_m, x_m)), F(gy_m, gx_m)\big) = 0, \end{cases}$$

where $\{x_m\}$ and $\{y_m\}$ are sequences in X such that

$$\begin{cases} \lim\limits_{m\to\infty} F(x_m, y_m) = \lim\limits_{m\to\infty} g(x_m) = x \\ \lim\limits_{m\to\infty} F(y_m, x_m) = \lim\limits_{m\to\infty} g(y_m) = y, \end{cases}$$

for some $x, y \in X$ are satisfied.

In [1], Abbas et al. introduced the following definition:

Definition 1.3.12. [1] The mappings $F : X^2 \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Recently, the idea of coupled fixed point is extended to higher dimensions by many authors (e.g. [25], [26], [70], [91], [98], [173]). In 2010, Samet and Vetro [174] firstly introduced the notion of fixed point of order $n \geq 3$, $n \in \mathbb{N}$, and proved some n-tupled fixed point results in ordered complete metric spaces, using a new concept of f-invariant set.

Definition 1.3.13. [174] Let X be a non-empty set and $F : X^n \to X$ a mapping $(n \geq 2)$. An element $(x^1, x^2, \dots, x^n) \in X^n$ is called a fixed point of n-order (or n-tupled fixed point) of the mapping F, if

$$\begin{cases} F(x^1, x^2, x^3, \dots, x^n) = x^1 \\ F(x^2, x^3, \dots, x^n, x^1) = x^2 \\ F(x^3, \dots, x^n, x^1, x^2) = x^3 \\ \qquad \vdots \\ F(x^n, x^1, x^2, \dots, x^{n-1}) = x^n. \end{cases}$$

In an attempt to extend the definition from X^2 to X^3, Berinde and Borcut [25] introduced the concept of tripled fixed point and utilize the same to prove some tripled fixed point theorems using mixed monotone property (see also [19, 163]). Here it can be pointed out that the notion of tripled fixed point due to Berinde and Borcut [25] is different from the one defined by Samet and Vetro [174] for $n = 3$ in the case of ordered metric spaces in order to keep the mixed monotone property working.

Definition 1.3.14. [25] Let $(X, \preceq)$ be a partially ordered set and $F : X^3 \to X$ a mapping. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$

$$\begin{aligned} &x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z) \\ &y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x, y_2, z) \preceq F(x, y_1, z) \\ &z_1, z_2 \in X,\ z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2). \end{aligned}$$

Definition 1.3.15. [25] An element $(x, y, z) \in X^3$ is called a tripled fixed point of a mapping $F : X^3 \to X$ if

$$F(x, y, z) = x,\ F(y, x, y) = y \ \text{ and } \ F(z, y, x) = z.$$

Following this trend, Karapinar [98] introduced the concept of quadrupled fixed point and utilize the same to prove some quadrupled fixed point theorems.

Definition 1.3.16. [98] Let X be a non-empty set and $F : X^4 \to X$ a mapping. An element $(x, y, z, w) \in X^4$ is called a quadrupled fixed point of the mapping F if

$$F(x, y, z, w) = x,\ F(y, z, w, x) = y,\ F(z, w, x, y) = z \text{ and } F(w, x, y, z) = w.$$

Definition 1.3.17. [98] Let $(X, \preceq)$ be an ordered set and $F : X^4 \to X$ a mapping. Then F is said to have mixed monotone property if for any $x, y, z, w \in X$, $F(x, y, z, w)$ is monotone non-decreasing in x and z and monotone non-increasing in y and w, *i.e.*, for

$$\begin{aligned} x_1, x_2 \in X,\ x_1 \preceq x_2 &\Rightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w) \\ y_1, y_2 \in X,\ y_1 \preceq y_2 &\Rightarrow F(x, y_2, z, w) \preceq F(x, y_1, z, w) \\ z_1, z_2 \in X,\ z_1 \preceq z_2 &\Rightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w) \\ w_1, w_2 \in X,\ w_1 \preceq w_2 &\Rightarrow F(x, y, z, w_2) \preceq F(x, y, z, w_1). \end{aligned}$$

Similar to coupled case, we can extend the tripled/quadrupled fixed points and mixed monotone property for a pair of mappings (F and g) by defining the concept of tripled/quadrupled coincidence points and mixed g-monotone property.

Very recently, Imdad et al. [91] generalized the idea of n-tupled fixed point by considering even-tupled coincidence point besides exploiting the idea of mixed g- monotone property on X^n and proved an even-tupled coincidence point theorem for non-linear ϕ-contraction mapping satisfying mixed g-monotone property. The definitions are as follows:

From now onwards, n stands for a general even natural number.

Definition 1.3.18 [91] Let X be a non-empty set and $F : X^n \to X$, $g : X \to X$ two mappings. An element $(x^1.x^2, ..., x^n) \in X^n$ is called an n-tupled coincidence point of the mappings F and g if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = gx^1 \\ F(x^2, x^3, ..., x^n, x^1) = gx^2 \\ F(x^3, ..., x^n, x^1, x^2) = gx^3 \\ \qquad\vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = gx^n. \end{cases}$$

In this case $(gx^1, gx^2, gx^3, ..., gx^n)$ is called an n-tupled point of coincidence.

If $g = I$ (identity mapping), then Definition 3.2.1 reduces to the definition of n-tupled fixed point.

Remark 1.3.1. Notice that, for n = 2; Definition 1.3.18 reduces to the definition of coupled coincidence point while on setting n = 4; it gives rise the definition of quadrupled coincidence point.

Definition 1.3.19. [91] Let $(X, \preceq)$ be an ordered set and $F : X^n \to X$, $g : X \to X$ two mappings. The mapping F is said to have the mixed g-monotone property if F is g-nondecreasing in its odd position arguments and g-nonincreasing in its even position arguments, that is, for any $x^1, x^2, x^3, ..., x^n \in X$

$$\begin{gathered} x_1^1, x_2^1 \in X,\ gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2. x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n) \\ x_1^2, x_2^2 \in X,\ gx_1^2 \preceq gx_2^2 \Rightarrow F(x^1, x_2^2, x^3, ..., x^n) \preceq F(x^1, x_1^2, x^3, ..., x^n) \\ x_1^3, x_2^3 \in X,\ gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n) \\ \vdots \\ x_1^n, x_2^n \in X,\ gx_1^n \preceq gx_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_2^n) \preceq F(x^1, x^2, x^3, ..., x_1^n). \end{gathered}$$

If $g = I$ (identity mapping) in Definition 1.3.19, then the mapping F is said to have the mixed monotone property.

Definition 1.3.20. [92] Let X be a non-empty set and $F : X^n \to X$, $g : X \to X$ two mappings. An element $(x^1, x^2, ..., x^n) \in X^n$ is called a common n-tupled fixed point of F and g if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = gx^1 = x^1 \\ F(x^2, x^3, ..., x^n, x^1) = gx^2 = x^2 \\ F(x^3, ..., x^n, x^1, x^2) = gx^3 = x^3 \\ \qquad\vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = gx^n = x^n. \end{cases}$$

Definition 1.3.21. [91] Let X be a non-empty set. Then we say that the mappings $F : X^n \to X$ and $g : X \to X$ are commutative if

$$F(gx^1, gx^2, ..., gx^n) = g\big(F(x^1, x^2, ..., x^n)\big) \text{ for all } x^1, x^2, ..., x^n \in X.$$

Definition 1.3.22. [8] Let $(X, d, \preceq)$ be an ordered metric space. We say that $(X, d, \preceq)$ has monotone convergence boundedness (abbreviated as MCB) property if it satisfies the following conditions:
(i) every non-decreasing convergent sequence $\{x_m\}$ in X is bounded above by its limit (as an upper bound), i.e.,

$$x_m \preceq x_{m+1} \text{ and } x_m \to x \text{ implies } x_m \preceq x, \text{ for all } m \geq 0;$$

(ii) every non-increasing convergent sequence $\{x_m\}$ in X is bounded below by its limit (as a lower bound), i.e.,

$$x_{m+1} \preceq x_m \text{ and } x_m \to x \text{ implies } x \preceq x_m, \text{ for all } m \geq 0.$$

Definition 1.3.23. [8] Let $(X, d, \preceq)$ be an ordered metric space. We say that $(X, d, \preceq)$ has g-MCB property if it satisfies the following conditions:
(i) g-image of every non-decreasing convergent sequence $\{x_m\}$ in X is bounded above by g-image of its limit (as an upper bound), i.e.,

$$x_m \preceq x_{m+1} \text{ and } x_m \to x \text{ implies } gx_m \preceq gx, \text{ for all } m \geq 0;$$

(ii) g-image of every non-increasing convergent sequence $\{x_m\}$ in X is bounded below by g-image of its limit (as a lower bound), i.e.,

$$x_{m+1} \preceq x_m \text{ and } x_m \to x \text{ implies } gx \preceq gx_m, \text{ for all } m \geq 0.$$

1.4 Some Basic Concepts and Results

In this section, we present some basic notations, definitions and results of functional analysis which will be used in the subsequent chapters.

Throughout the thesis, unless otherwise stated, X denotes a real Hilbert space, E denotes a real Banach space, E^* denotes the topological dual of E, 2^E denotes the power set of E, $CB(E)$ denotes the family of all nonempty, closed and bounded subsets of E. If there is no confusion likely to occur, we denote the norm of Hilbert space X, Banach space E and their dual spaces by $\|\cdot\|$, and denote the inner product of X and the duality pairing between E and E^* by $\langle\cdot,\cdot\rangle$.

Definition 1.4.1. [36, 81] Let K be a non-empty and convex subset of X and $f : K \to \mathbb{R}$. Then

(i) f is said to be convex if, for any $x, y \in K$ and for any $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y);$$

(ii) f is said to be lower semicontinuous on K if, for every $\alpha \in \mathbb{R}$, the set $\{x \in K : f(x) \leq \alpha\}$ is closed in K;

(iii) f is said to be concave if $(-f)$ is convex;

(iv) f is said to be upper semicontinuous on K if $(-f)$ is lower semicontinuous on K.

Note that the concept of lower semicontinuity and upper semicontinuity of set-valued mapping on Banach space can be defined in the same way as defined in Definition 1.4.1.

Definition 1.4.2. [20] A proper and convex functional $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be subdifferentiable at a point $x \in X$, if there exists a point $f^* \in X^*$ such that

$$\varphi(y) - \varphi(x) \geq \langle f^*, y - x\rangle, \quad \forall y \in X,$$

where f^* is called a subgradient of φ at x. The set of all subgradients of φ at x is denoted by $\partial\varphi(x)$. The mapping $\partial\varphi : X \to 2^{X^*}$ defined by

$$\partial\varphi(x) = \{f^* \in H^* : \varphi(y) - \varphi(x) \geq \langle f^*, y - x\rangle, \quad \forall y \in X\},$$

is called subdifferential of φ at x.

Theorem 1.4.1. [20] If $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a convex functional, then its subdifferential $\partial\varphi$ is monotone.

Definition 1.4.3. [138] The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(E)$ is defined by

$$\mathcal{D}(P,Q) = max\left\{\sup_{x\in P} d(x,Q),\ \sup_{y\in Q} d(P,y)\right\},\quad \text{for all } P,\ Q\in \text{CB}(E),$$

where $d(x,Q) = \inf_{y\in Q}\|x-y\|$ and $d(P,y) = \inf_{x\in P}\|x-y\|$.

Theorem 1.4.2. [138] Let X be a complete metric space. Suppose that $F : X \to CB(X)$ satisfies

$$\mathcal{D}\big(F(x),F(y)\big) \leq \alpha d(x,y),\ \forall x,y\in X,$$

where $\alpha\in[0,1)$ is a constant. Then the mapping F has a fixed point in X.

Definition 1.4.4. [37] A set-valued mapping $T : E \longrightarrow CB(E)$ is said to be $\mathcal{D}$-Lipschitz continuous if there exists a constant $l_T > 0$ such that

$$\mathcal{D}(T(x),T(y)) \leq l_T\|x-y\|,\quad \forall x,y\in X.$$

Theorem 1.4.3 [183] Let E be a Banach space and F is set-valued contraction mapping on E into itself. Then F has a fixed point.

Definition 1.4.5. [102, 193] A single-valued mapping $A : X\to X$ is said to be

(i) monotone, if
$$\langle A(x)-A(y),x-y\rangle \geq 0,\quad \forall\ x,y\in X;$$

(ii) strictly monotone, if
$$\langle A(x)-A(y),x-y\rangle > 0,\quad \forall\ x,y\in X,$$
and equality holds if and only if $x=y$;

(iii) strongly monotone, if there exists a constant $\delta_1>0$ such that
$$\langle A(x)-A(y),x-y\rangle \geq \delta_1\|x-y\|^2,\quad \forall\ x,y\in X;$$

(iv) relaxed monotone, if there exists a constant $\epsilon_1>0$ such that
$$\langle A(x)-A(y),x-y\rangle \geq (-\epsilon_1)\|x-y\|^2,\quad \forall\ x,y\in X;$$

(v) cocoercive, if there exists a constant $\mu_1>0$ such that
$$\langle A(x)-A(y),x-y\rangle \geq \mu_1\|A(x)-A(y)\|^2,\quad \forall\ x,y\in X;$$

(vi) relaxed cocoercive, if there exists a constant $\gamma_1 > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq (-\gamma_1)||A(x) - A(y)||^2, \quad \forall \ x, y \in X;$$

(vii) λ_A-Lipschitz continuous, if there exists a constant $\lambda_A > 0$ such that

$$||A(x) - A(y)|| \leq \lambda_A||x - y||, \quad \forall \ x, y \in X;$$

(viii) α_A-expansive, if there exists a constant $\alpha_A > 0$ such that

$$||A(x) - A(y)|| \geq \alpha_A||x - y||, \quad \forall \ x, y \in X;$$

If $\alpha_A = 1$, then it is expansive.

Definition 1.4.6. [**10, 18**] A set-valued mapping $M : X \to 2^X$ is said to be

(i) monotone, if

$$\langle u - v, \ x - y\rangle \geq 0, \quad \forall x, y \in X, \ u \in M(x), \ v \in M(y);$$

(ii) strongly monotone, if there exists a constant $\delta_2 > 0$ such that

$$\langle u - v, \ x - y\rangle \geq \delta_2||x - y||^2, \quad \forall x, y \in X, \ u \in M(x), \ v \in M(y);$$

(iii) relaxed monotone, if there exists a constant $\epsilon_2 > 0$ such that

$$\langle u - v, \ x - y\rangle \geq (-\epsilon_2)||x - y||^2, \quad \forall x, y \in X, \ u \in M(x), \ v \in M(y);$$

(iv) cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle u - v, \ x - y\rangle \geq \mu_2||u - v||^2, \quad \forall x, y \in X, \ u \in M(x), \ v \in M(y);$$

(v) maximal monotone if M is monotone and $(I + \rho M)(X) = X$ for some constant $\rho > 0$, where I denotes the identity operator on X.

Definition 1.4.7. [**18, 59, 207**] Let $M : X \to 2^X$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping

$$J_\rho^M(x) = (I + \rho M)^{-1}(x), \quad \forall x \in X,$$

is called the proximal-point (resolvent) mapping of M, where I stands for identity mapping on X.

Remark 1.4.1. [18]

(i) The proximal-point mapping J_ρ^M is single-valued and nonexpansive, *i.e.*,

$$\|J_\rho^M(x) - J_\rho^M(x)\| \leq \|x - y\|, \ \forall x, y \in X.$$

(ii) Since the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a maximal monotone operator, it follows that the proximal point mapping of $\partial\varphi$

$$J_\rho^{\partial\varphi}(x) = (I + \rho\partial\varphi)^{-1}(x), \quad \forall x \in X,$$

is also single-valued and nonexpansive.

Definition 1.4.8. [60, 83] Let $\eta : X \times X \to X$ and $H : X \to X$ be the single-valued mappings and $M : X \to 2^X$ be a set-valued mapping. Then

(i) η is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \ \|x - y\| \ \ \forall x, \ y \in X;$$

(ii) M is said to be η-monotone, if

$$\langle u - v, \ \eta(x, y)\rangle \ \geq \ 0 \ \ \forall x, \ y \in X, \ u \in M(x), \ v \in M(y);$$

(iii) M is said to be strictly η-monotone if, M is η-monotone and equality holds if and only if $x = y$;

(iv) M is said to be strongly η-monotone, if there exists a constant $\chi > 0$ such that

$$\langle u - v, \ \eta(x, y)\rangle \ \geq \ \chi \ \|x - y\|^q \ \ \forall x, \ y \in X, \ u \in M(x), \ v \in M(y);$$

(v) M is said to be maximal η-monotone, if M is η-monotone and $(I + \rho \ M)(X) = X$ for all $\rho > 0$, where I denotes the identity operator on X;

(vi) M is said to be H-monotone, if M is monotone and $(H + \rho \ M)(X) = X$ for all $\rho > 0$;

(vii) M is said to be (H, η)-monotone, if M is η-monotone and $(H + \rho \ M)(X) = X$ for all $\rho > 0$;

Definition 1.4.9. [47] A continuous and strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a gauge (weight) function.

Definition 1.4.10. [47] Let E be a Banach space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a gauge function. A set-valued mapping $J : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|;\ \|f\| = \varphi(\|x\|)\}, \text{ for all } x \in E,$$

is called the duality mapping with gauge function φ.

A selection of the duality mapping J is a single-valued mapping $j : E \to E^*$ satisfying $jx \in J(x)$ for each $x \in E$. Furthermore, if $\varphi(\|x\|) = \|x\|$, $\forall x \in E$, then J is called normalized duality mapping.

Remark 1.4.2. [47] If $E \equiv X$, then normalized duality mapping J becomes identity mapping.

Theorem 1.4.4. [10, 47] Let J be a duality mapping associated with a weight φ, then

(a) J is monotone, *i.e.*,

$$\langle x^* - y^*, x - y\rangle \ \geq\ 0,\ \ \forall x, y \in E,\ x^* \in J(x),\ y^* \in J(y);$$

(b) $J(-x) = -J(x), \ \ \forall x \in E$;

(c) $J(\lambda x) \ = \ \frac{\varphi(\lambda\|x\|)}{\varphi\|x\|} J(x), \ \ \forall x \in E$ and $\lambda > 0$.

Theorem 1.4.5. [160] Let E be a Banach space and $J : E \to 2^{E^*}$ a normalized duality mapping. Then, for any $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \ \forall j(x + y) \in J(x + y).$$

Definition 1.4.11. [47, 205] A Banach space E is called smooth if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$, defined by

$$\rho_E(\tau) = \sup\left\{\frac{(\|x + y\| + \|x - y\|)}{2} - 1 :\ x, y \in E, \|x\| = 1, \|y\| = \tau\right\}.$$

Definition 1.4.12. [47, 205] The Banach space E is said to be uniformly smooth if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Theorem 1.4.6. [47] A Banach space is smooth if and only if each duality mapping J of weight φ is a single-valued.

Theorem 1.4.7. [47] If E is a uniformly smooth Banach space, then E is reflexive.

Theorem 1.4.8. [**41, 54, 68**] Let E be a uniformly smooth Banach space and $J : E \to E^*$ the normalized duality mapping. Then for all $x, y \in E$, we have

(a) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle$;

(b) $\langle x-y, J(x)-J(y)\rangle \leq 2d^2\rho_E(4\|x-y\|/d)$, where $d = \sqrt{(\|x\|^2+\|y\|^2)/2}$.

Definition 1.4.13. [**204**] For $q > 1$, a mapping $J_q : E \to 2^{E^*}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q, ||x||^{q-1} = ||f||\}, \ \forall x \in E.$$

Remark 1.4.3. [**204**] In particular, J_2 is the usual normalized duality mapping on E. Also, we have

$$J_q(x) = ||x||^{q-2}J_2(x), \quad \forall x(\neq 0) \in E.$$

Definition 1.4.14. [**204, 205**] The Banach space E is said to be q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^q, \quad \tau \in [0, \infty).$$

Remark 1.4.4. [**204**]

(i) It is well known (see for example [204]) that

$$L_q \text{ (or } l_q) \text{ is} \begin{cases} q\text{-uniformly smooth}, & \text{if } 1 < q \leq 2, \\ \\ 2\text{-uniformly smooth}, & \text{if } q \geq 2. \end{cases}$$

(ii) If E is uniformly smooth, J_q becomes single-valued.

Lemma 1.4.1. [**204**] Let $q > 1$ be a real number and E a smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,

$$||x+y||^q \leq ||x||^q + q\langle y, J_q(x)\rangle + c_q||y||^q.$$

Definition 1.4.15. [**209, 210**] Let E be a q-uniformly smooth Banach space and $A : E \to E$ a single-valued mapping. Then A is said to be

(i) accretive, if

$$\langle A(x) - A(y),\ J_q(x-y)\rangle \geq 0, \text{ for all } x, y \in E;$$

(ii) strictly accretive, if

$$\langle A(x) - A(y),\ J_q(x-y)\rangle > 0, \text{ for all } x, y \in E,$$

and equality holds if and only if $x = y$;

(iii) strongly accretive, if there exists a constant $\delta_3 > 0$ such that

$$\langle A(x) - A(y),\ J_q(x-y)\rangle \geq \delta_3\|x-y\|^q, \text{ for all } x, y \in E;$$

(iv) relaxed accretive, if there exists a constant $\epsilon_3 > 0$ such that

$$\langle A(x) - A(y),\ J_q(x-y)\rangle \geq (-\epsilon_3)\|x-y\|^q, \text{ for all } x, y \in E.$$

Definition 1.4.16. [209, 210] Let E be a q-uniformly smooth Banach space. Let $\eta : E \times E \to E,\ H,\ A,\ B:\ E \to E$ be single-valued mappings and $M:\ E \to 2^E$ a set-valued mapping. Then

(i) M is said to be accretive, if

$$\langle u - v,\ J_q(x-y)\rangle\ \geq\ 0\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(ii) M is said to be strongly accretive, if there exists a constant $\delta_4 > 0$ such that

$$\langle u - v,\ J_q(x-y)\rangle\ \geq\ \delta_4\|x-y\|^q\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(iii) M is said to be relaxed accretive, if there exists a constant $\epsilon_4 > 0$ such that

$$\langle u - v,\ J_q(x-y)\rangle\ \geq\ (-\epsilon_4)\|x-y\|^q\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(iv) M is said to be η-accretive, if

$$\langle u - v,\ J_q(\eta(x,y))\rangle\ \geq\ 0\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(v) M is said to be strictly η-accretive, if M is η-accretive and equality holds if and only if $x = y$;

(vi) M is said to be strongly η-accretive, if there exists a constant $\delta_5 > 0$ such that

$$\langle u - v,\ J_q(\eta(x,y))\rangle\ \geq\ \delta_5\ \|x-y\|^q\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(vii) M is said to be relaxed η-accretive, if there exists a constant $\epsilon_5 > 0$ such that

$$\langle u - v,\ J_q(\eta(x, y))\rangle \ \geq\ (-\epsilon_5)\ \|x - y\|^q\ \ \forall x,\ y \in E,\ u \in M(x),\ v \in M(y);$$

(viii) M is said to be m-accretive, if M is accretive and $(I + \rho\, M)(E) = E$ for all $\rho > 0$, where I denotes the identity operator on E;

(ix) M is said to be generalized m-accretive, if M is η-accretive and $(I + \rho\, M)(E) = E$ for all $\rho > 0$;

(x) M is said to be H-accretive, if M is accretive and $(H + \rho\ M)(E) = E$ for all $\rho > 0$;

(xi) M is said to be (H, η)-accretive, if M is η-accretive and $(H + \rho\ M)(E) = E$ for all $\rho > 0$;

(xii) M is said to be (A, η)-accretive, if M is relaxed η-accretive and $(A + \rho\, M)(E) = E$ for all $\rho > 0$.

Definition 1.4.17. [209] Let E be a q-uniformly smooth Banach space. Let $A,\ B:\ E \to E$, $H: E \times E \to E$ be three single-valued mappings and $M:\ E \to 2^E$ a set-valued mapping. Then M is said to be $H(.,.)$-accretive with respect to A and B if M is accretive and $(H(.,.) + \rho\ M)(E) = E$ for all $\rho > 0$.

Example 1.4.1. [209] Let $E\ =\ \mathbb{R}^2\ =\ (-\infty, +\infty) \times (-\infty, +\infty)$ and $q = 2$. Let $A, B: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax\ =\ (3x_1,\ 3x_2),\ By\ =\ (-y_1, -y_2),\ \ \forall\, x = (x_1,\ x_2), y = (y_1,\ y_2) \in \mathbb{R}^2.$$

Suppose that $H:\ \ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$H(Ax, By)\ =\ Ax\ +\ By\ \ \forall\, x, y \in \mathbb{R}^2.$$

Then, $H(A, B)$ is 3-strongly accretive with respect to A and 1-relaxed accretive with respect to B.

Example 1.4.2. [209] Let $E\ =\ \mathbb{R}\ =\ (-\infty, +\infty)$, $q = 2$ and $Mx =\ 2x$ for all $x \in \mathbb{R}$. Then M is 1-strongly accretive.

Example 1.4.3. [209] Let $E\ =\ \mathbb{R}$, $Mx = x$ and $Hx =\ x^3$ for all $x \in \mathbb{R}$. Then, M is an m-accretive operator and also an H-accretive operator. However, if $Hx =\ x^2$ for all $x \in \mathbb{R}$, then M is not an H-accretive operator. In fact, $(H + M)(x) = x + x^2 \geq -\frac{1}{4}$, so $(H + M)$ is not surjective.

Example 1.4.4. [209] Let $E = \mathbb{R}^2$ and $q = 2$. Let $M(x) = (\sin x_2, 0)$, $Hx = (x_1^3, x_2^3)$ and $\eta(x, y) = (\sin x_2 - \sin y_2, \sin y_1 - \sin x_1)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then M is an η-accretive mapping. Furthermore, M is (H, η)-accretive operator since $(H + \rho M)(\mathbb{R}^2) = \mathbb{R}^2$ for any $\rho > 0$.

Example 1.4.5. [209] Let $E = \mathbb{R}^2$ and $q = 2$. Let $M(x) = (\sin x_2, \pi)$ and $\eta(x, y) = (\sin y_1 - \sin x_1, 0)$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then M is 1-relaxed η-accretive mapping. If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined the same as the mapping H in the above example, then M is an (A, η)-accretive mapping.

Definition 1.4.18. Let E be a q-uniformaly smooth Banach space and $\eta : E \times E \to E$ a single-valued mapping. Then $A : X \to X$ is said to be

(i) η-accretive, if

$$\langle A(x) - A(y), J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) strictly η-accretive, if A is η-accretive and equality holds if and only if $x = y$;

(iii) δ_5-η-strongly accretive, if there exists a constant $\delta_5 > 0$ such that

$$\langle A(x) - A(y), J_q(\eta(x, y))\rangle \geq \delta_5 \|x - y\|^q, \quad \forall x, y \in X;$$

(iii) ϵ_5-η-relaxed accretive, if there exists a constant $\epsilon_5 > 0$ such that

$$\langle A(x) - A(y), J_q(\eta(x, y))\rangle \geq \epsilon_5 \|x - y\|^q, \quad \forall x, y \in X;$$

Definition 1.4.19. [202] Let E be a q-uniformly smooth Banach space. Let $A, B : E \to E$, H, $\eta : E \times E \to E$ be the single-valued mappings and $M : E \to 2^E$ a set-valued mapping. Then M is said to be $(H(.,.)$-$\eta)$-accretive with respect to A and B if M is m-η-relaxed accretive and $(H(.,.) + \rho M)(E) = E$ for all $\rho > 0$.

1.5 Literature of Variational Inequalities

Theory of variational inequalities was initiated independently by G. Stampacchia [192] and G. Fichera [65] in the early 1960's to study the problems in potential theory and elasticity, respectively. The first general theorem for the existence and uniqueness of solution of variational inequality was proved by Lions and Stampacchia [122] in 1967. Since then variational inequalities have been extended and generalized in several directions using novel and innovative techniques both for their own sake and for applications.

It has been shown that variational inequality theory provides the natural, direct, unified and efficient framework for the general treatment of a wide class of unrelated linear and nonlinear problems arising in fluid flow through porous media, elasticity, transformation, economics, optimization, structural analysis, applied and engineering sciences (see [14, 15, 16, 17, 18, 20, 23, 24, 47, 54, 51, 58, 59, 66, 67, 68, 113, 153, 207]).

In this section, we give brief survey of some classes of variational inequalities, variational inclusions, systems of variational inequalities and systems of variational inclusions.

I. Variational Inequalities and Variational Inclusions:

Let K be a nonempty, closed and convex subset of a Hilbert space X and $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ a bilinear form.

Problem 1.5.1. For given $f \in X^*$, find $x \in K$ such that

$$a(x, y - x) \geq \langle f, y - x\rangle, \quad \forall y \in K. \tag{1.5.1}$$

The inequality (1.5.1) is termed as variational inequality which characterizes the classical Signorini problem of elastostatistics, that is, the analysis of a linear elastic body in contact with a rigid frictionless foundation. This problem was investigated and studied by Lions and Stampacchia [122] by using the projection technique.

If the bilinear form is continuous, then by Riesz-Fréchet theorem [172], we have

$$a(x, y) = \langle A(x), y\rangle, \ \forall x, y \in X,$$

where $A : X \to X^*$ is a continuous linear operator. Then Problem 1.5.1 is equivalent to the following problem:

Problem 1.5.2. Find $x \in K$ such that

$$\langle A(x), y - x\rangle \geq \langle f, y - x\rangle, \ \forall y \in K. \tag{1.5.2}$$

If $f \equiv 0 \in X^*$, then (1.5.2) reduces to the following classical variational inequality problem:

Problem 1.5.3. Find $x \in K$ such that

$$\langle A(x), y - x\rangle \geq 0, \ \forall y \in K. \tag{1.5.3}$$

In the variational inequality formulation, the underlying convex set K does not depend upon the solution. In many important applications, the convex set K also depends implicitly on the solution. In this case, variational inequality (1.5.1) is known as quasi-variational inequality which arises in the study of impulse control theory and decision science (see [23, 24]). Quasi-variational inequality was introduced and studied by Bensoussan, Goursat and Lions [22]. To be more precise, given a set-valued mapping $K : x \to K(x)$, which associates a nonempty, closed and convex subset $K(x)$ of X for each $x \in X$, a typical quasi-variational inequality problem is:

Problem 1.5.4. Find $x \in K(x)$ such that

$$a(x, y - x) \geq \langle f, y - x \rangle, \quad \forall y \in K(x). \tag{1.5.4}$$

In many important applications, see for example Baiocchi and Capelo [20], Bensoussan and Lions [23] and Mosco [136], the underlying set $K(x)$ is of the following form:

$$K(x) = C + m(x),$$

where $m : X \to X$ is a nonlinear mapping and C is a nonempty, closed and convex subset in X. Note that if m is a zero mapping, then Problem 1.5.4 is same as Problem 1.5.1.

Further, since the general problem of equilibrium of elastic bodies in contact with rigid foundation on which frictional forces are developed is one of the most difficult problems in solid mechanics. Duvaut and Lions [58] investigated the following variational inequality problem with friction:

Problem 1.5.5. For $f \in X^*$, find $x \in K$ such that

$$a(x, y - x) + \phi(y) - \phi(x) \geq \langle f, y - x \rangle, \; \forall y \in K, \tag{1.5.5}$$

where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. Problem 1.5.5 characterizes the classical Signorini problem with frictional force. The existence of its solution has been proved by Glowinski *et al.* [67] and Nečas *et al.* [142].

The complete study of boundary value problem arising in the formulation of Signorini problem with friction is an interesting problem both in mechanics and mathematical points of view. A generalization of the Problem 1.5.5 is the following:

Problem 1.5.6. Given $f \in X^*$, find $x \in K$ such that

$$a(x, y - x) + \psi(x, y) - \psi(x, x) \geq \langle f, y - x \rangle, \quad \forall y \in K, \tag{1.5.6}$$

where $\psi : X \times X \to \mathbb{R}$ is an appropriate nonlinear form. This type of problems have been studied in Duvaut and Lions [58], Kikuchi and Oden [112], Noor [146]. The Problem

1.5.6 characterizes the fluid flow through porous media and Signorini problems with non-local frictions. For physical and mathematical formulation of the inequality (1.5.6), see Oden and Pires [153]. For related work, see also Baiocchi and Capelo [20] and Crank [51].

In 1975, Noor [145] extended the Problem 1.5.1 to study a class of mildly nonlinear elliptic boundary value problems having constraints. Given nonlinear operators $T, A : X \to X^*$, Noor [145] considered the following problem:

Problem 1.5.7. Find $x \in K$ such that

$$\langle T(x), y - x\rangle \geq \langle A(x), y - x\rangle, \quad \forall y \in K. \tag{1.5.7}$$

Then inequality (1.5.7) is known as mildly nonlinear variational inequality.

Problem 1.5.8. Find $x \in K$ such that

$$\langle T(x), y - x\rangle + \phi(y) - \phi(x) \geq \langle A(x), y - x\rangle, \quad \forall y \in K, \tag{1.5.8}$$

where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous functional. Problem 1.5.8 has been studied by Siddiqi *et al.* [184] in the setting of Banach space.

We note that the above mentioned classes of variational and quasi-variational inequalities are applicable in the study of the boundary value problems of even order only. In 1988, Noor [147] introduced and studied the following variational inequality problem which characterizes the odd order boundary value problems:

Problem 1.5.9. Find $x \in X$ such that $g(x) \in K$ and

$$\langle T(x), g(y) - g(x)\rangle \geq 0, \quad \forall g(y) \in K, \tag{1.5.9}$$

where $T, g : X \to X$. Problem 1.5.9 is called general variational inequality problem. The corresponding general quasi-variational inequality problem to Problem 1.5.9 is as follows:

Problem 1.5.10. Find $x \in X$ such that $g(x) \in K(x)$ and

$$\langle T(x), g(y) - g(x)\rangle \geq 0, \quad \forall g(y) \in K(x), \tag{1.5.10}$$

which has been considered and studied by Noor [148].

Let $T, A : X \to 2^X$ be two set-valued mappings, $g : X \to X$ a single-valued mapping and $K : X \to 2^X$ a set-valued mapping such that for each $x \in X$, $K(x)$ is a nonempty, closed and convex subset of X. Let $N : X \times X \to X$ be a nonlinear single-valued mapping, then the following problem is considered and studied by Noor [152]:

Problem 1.5.11. Find $x \in E$, $u \in T(x)$, $v \in A(x)$, $g(x) \in K(x)$ such that

$$\langle g(x) + N(u, v), y - g(x)\rangle \ \geq \ 0, \ \ \forall y \in K(x), \tag{1.5.11}$$

which is called set-valued implicit quasi-variational inequality problem. For applications of Problem 1.5.11, see [152] and the references cited therein.

Variational-like inequality is a generalization of variational inequality, which is introduced by Parida *et al.* [157] in 1989. They studied the existence of solution of variational inequality in n-dimensional Euclidean space and gave its application to nonconvex mathematical programming problems.

Let K be a closed convex set in $\mathbb{R}^n$, $T : K \to \mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ are single-valued continuous mappings.

Problem 1.5.12. Find $x \in K$ such that

$$\langle T(x), \eta(y, x)\rangle \geq 0, \ \forall y \in K. \tag{1.5.12}$$

If $\eta(y, x) = y - x$, then variational-like inequality (1.5.12) reduces to variational inequality (1.5.3). Now, we give some generalizations of Problem 1.5.12.

Problem 1.5.13. Find $x \in K$ such that

$$\langle T(x), \eta(y, x)\rangle \geq \phi(x) - \phi(y), \ \forall y \in K. \tag{1.5.13}$$

Problem 1.5.13 has been considered and studied by Dien [55] in $\mathbb{R}^n$. It has been further considered and studied by Siddiqi *et al.* [185] in the setting of reflexive Banach spaces and topological vector spaces with or without convexity assumptions.

If, in a variational inequality, similar to inequalities (1.5.6) or (1.5.8), the underlying region is the entire space X instead of the nonempty, closed and convex subset K of X and the functional $\psi(\cdot, \cdot)$ (or $\phi(\cdot)$) is subdifferentiable, then such type of variational inequality is called variational inclusion. Here, we give some classes of variational inclusion problems:

Let $A, B, g : X \to X$ and $N : X \times X \to X$ be nonlinear mappings and $M : X \to 2^X$ a set-valued maximal monotone mapping. Let $S, T : X \to 2^X$ be set-valued mappings and $\partial\phi$ is the subdifferential of a proper, convex and lower semicontinuous functional $\phi : X \to \mathbb{R} \cup \{+\infty\}$. Let $A : X \to X^*$, $f \in X^*$.

Problem 1.5.14 Find $x \in X$ such that

$$\langle Ax - f, y - x\rangle \geq \varphi(x) - \varphi(y) \ \ \forall y \in X. \tag{1.5.14}$$

Problem 1.5.14 is called variational inclusion problem and has been introduced and studied by Brézis [27] in the setting of reflexive Banach space.

Problem 1.5.15. Find $x \in X$ such that $g(x) \in dom(\partial\phi)$ and

$$\langle A(x) - B(x), y - g(x)\rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in X, \tag{1.5.15}$$

where $dom(\partial\phi)$ denotes the domain of $\partial\phi$.

Variational inclusion problem (1.5.15) has been introduced and studied by Hassouni and Moudafi [79].

In addition, if $\phi \equiv \delta_K$, where δ_K is the indicator function of a nonempty, closed and convex set K in X, defined as:

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K. \\ +\infty, & \text{if } x \notin K. \end{cases}$$

Then Problem 1.5.15 reduces to a variational inequality problem similar to Problem 1.5.7.

Problem 1.5.16. Find $x \in X$, $u \in S(x)$, $v \in T(x)$ such that $(g - m)(x) \in dom(\partial\phi)$ and

$$\langle u - v, y - (g - m)(x)\rangle \geq \phi((g - m)(x)) - \phi(v), \quad \forall y \in X, \tag{1.5.16}$$

where $(g - m)$ is defined as

$$(g - m)(x) = g(x) - m(x), \quad \forall x \in X.$$

Problem 1.5.16 is called generalized quasi-variational inclusion problem and has been introduced and studied by Kazmi [104].

Problem 1.5.17. Find $x \in X$ such that

$$\Theta \in (A - B)(x) + M(g(x)), \tag{1.5.17}$$

where Θ is the zero element in X. Problem 1.5.17 has been studied by Adly [3].

Clearly for $M \equiv \partial\phi$, Problem 1.5.17 reduces to Problem 1.5.15.

Problem 1.5.18. Find $x \in X$, $u \in T(x)$ and $v \in S(x)$ such that

$$\Theta \in N(u, v) + \lambda M(g(x)), \tag{1.5.18}$$

where $\lambda > 0$ is a constant. Problem 1.5.18 has been studied by Noor *et al.* [151].

II. Systems of Variational Inequalities and Variational Inclusions:

In recent past, system of variational inequalities are used as tools to solve various equilibrium problems e.g., Nash equilibrium problem, Spatial equilibrium problem, general equilibrium programming problem, problems from operations research, economics, game theory, mathematical physics and other areas (see [15, 103, 116, 155]) and references therein. In 1985, Pang [155] uniformly modeled these equilibrium problems in the form of a variational inequality defined on a product of sets. He decomposed the original variational inequality into a system of variational inequalities, which are easy to solve, to establish some solution methods for finding the approximate solutions of above mentioned equilibrium problems. Later, it is found that these two problems, variational inequality defined on a product of sets and system of variational inequalities, are equivalent. Thereafter, a number of researchers introduced and studied various classes of systems of variational inequalities (inclusions) (see [6, 31, 56, 80, 108, 110, 158, 199, 201, 202, 203, 206]). Here, we give some classes of systems of variational inequalities (inclusions).

Let K be a nonempty, closed and convex subset of Hilbert space X and $T : K \times K \to X$ a nonlinear mapping on K.

Problem 1.5.19. Find $x_1, x_2 \in K$ such that

$$\begin{cases} \langle \rho T(x_2, x_1) + x_1 - x_2, a - x_1 \rangle \geq 0, \quad \forall a \in K, \\ \\ \langle \lambda T(x_1, x_2) + x_2 - x_1, a - x_2 \rangle \geq 0, \quad \forall a \in K \end{cases} \tag{1.5.19}$$

where $\rho > 0, \lambda > 0$ are some cosntants. Problem (1.5.19) is called system of variational inequalities which has been introduced and studied by Verma [196, 197].

For each $i = 1, 2$, let K_i be a nonempty closed and convex subset of Hilbert space X_i, $F_i : X_1 \times X_2 \to X_i$ the single-valued mappings.

Problem 1.5.20. Find $(x_1, x_2) \in K_1 \times K_2$ such that

$$\begin{cases} \langle F_1(x_1, x_2), a - x_1 \rangle \geq 0, \quad \forall a \in K_1, \\ \\ \langle F_2(x_1, x_2), b - x_2 \rangle \geq 0, \quad \forall b \in K_2 \end{cases} \tag{1.5.20}$$

The system of variational inequalities (1.5.20) has been introduced and studied by Kassay *et al.* [103].

For each $i = 1, 2$, let K_i be a nonempty closed and convex subset of Hilbert space

X_i, $F_i : X_1 \times X_2 \to X_i$, $H_i : X_i \to X_i$, and $\eta : X \times X \to X$ are single-valued mappings. Let $M_i : X_i \to 2^{X_i}$ be an (H_i, η)-monotone operator.

Problem 1.5.21. Find $(x_1, x_2) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in F_1(x_1, x_2) + M_1(x_1), \\ \\ 0 \in F_2(x_1, x_2) + M_2(x_2). \end{cases} \tag{1.5.21}$$

This system of variational inclusions has been introduced and studied by Fang *et al.* [63].

For each $i = 1, 2$, let X_i be real Hilbert spaces, $F_i : X_1 \times X_2 \to X_i$, $p_i : X_i \to X_i$ are single-valued mappings and $S_i : X_i \to 2^{X_i}$, the multi-valued mappings. For each $i = 1, 2$, let $A_i : X_i \to X_i$, $M_i : X_i \to 2^{X_i}$ and $f_i : X_i \to X_i$ be nonlinear mappings with $f_i(X_i) \cap Dom(M_i) \neq \emptyset$, respectively.

Problem 1.5.22. Find $(x_1, x_2) \in X_1 \times X_2$, $u_1 \in S_1(x_1)$, $u_2 \in S_2(x_2)$ such that

$$\begin{cases} 0 \in F_1(p_1(x_1), u_2) + M_1(f_1(x_1)), \\ \\ 0 \in F_2(u_1, p_2(x_2)) + M_2(f_2(x_2)). \end{cases} \tag{1.5.22}$$

This system of variational inclusions has been introduced and studied by Lan *et al.* [121].

Chapter 2

Some Common Fixed Point Theorems in Menger PM and PQM-Spaces

2.1 Introduction

In 1942, Menger [128] introduced the notion of probabilistic metric spaces (in short, PM-spaces) as a generalization of metric spaces. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations wherein we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. The study of probabilistic metric space received much attention after the pioneering work of Schweizer and Sklar [178] (also see [40, 179]). In 1972, Sehgal and Bharucha-Reid [181] obtained a generalization of the Banach contraction principle on a complete Menger PM-space, which proved a turning point for the development of fixed point theory in a Menger PM-space. Thereafter, various fixed point theorems in PM-spaces have been obtained (e.g. [46, 73, 90, 115, 132, 167, 170, 188]).

In 1989, Kent and Richardson [111] introduced the class of probabilistic quasi metric spaces (in short, PQM-spaces) and proved some common fixed point theorems. The theory of probabilistic quasi metric spaces can be used as an efficient tool to solve so many problems belonging to several domains, namely: theoretical computer science, approximation theory and topological algebra (see, for instance [72, 118, 133]). Fixed

The contents of this chapter are based on two research papers. One of them has been published in Jour. of Operators, 2013:391474, (2013) while the other one is published in Jour. Nonlinear Anal. Appl., 2014:jnaa-00142, (2014).

point theorems in PQM-spaces have appeared in [39, 130, 131, 133, 171, 156, 176, 180].

With a view to improve commutativity conditions in fixed point theorems, Sessa [182] introduced the concept of weakly commuting mappings. Inspired by this concept, Jungck [95] weakened the notion of weak commutativity by introducing the compatible mappings. Further, Jungck and Rhoades [96] introduced the notion of weak compatibility which is the most general among all commutativity concepts. In 2002, Lu [124] introduced the concept of the converse commuting mappings as a reverse process of weakly compatible mappings and proved common fixed point theorems in metric spaces. Many authors formulated the definitions of weakly commuting pair [186], compatibility [132] and weakly compatible pair [189] in the framework of probabilistic setting and proved several fixed point results.

Section 2.2 contains preliminaries which include some basic notions and core results. The details of implicit functions due to Singh and Jain [189] as well as Imdad and Ali [89] will be presented in Section 2.3. In Section 2.4, we prove some common fixed point theorems for two pairs of converse commuting mappings in Menger PM-spaces using the implicit functions discussed in Section 2.3 while Section 2.5 is devoted to some illustrative examples to demonstrate the results proved in Section 2.4. In Section 2.6, we obtain a common fixed point theorem for a pair of finite number of self-mappings in Menger PQM-space using weak compatibility.

2.2 Preliminaries

In this section, we collect the background material to make our presentation as self contained as possible.

Definition 2.2.1. [178] A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:
(i) Δ is commutative and associative;
(ii) $\Delta(a, 1) = a$ for all $a \in [0,1]$;
(iii) $\Delta(a, b) \leq \Delta(c, d)$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

The following are some examples of basic t-norms:
(i) $\Delta_M(a, b) = \min \{a, b\}$;
(ii) $\Delta_P(a, b) = ab$;
(iii) $\Delta_L(a, b) = \max \{a + b - 1, 0\}$.

Each t-norm Δ can be extended [114] (by associativity) in a unique way by taking $(x_1, ..., x_n) \in [0,1]^n$ $(n \in \mathbb{N})$ the values $T^1(x_1, x_2) = T(x_1, x_2)$ and $T^n(x_1, ..., x_{n+1}) =$

$T(T^{n-1}(x_1, ..., x_n), x_{n+1})$, for $n \geq 2$ and $x_i \in [0, 1]$, $\forall\, i \in \{1, 2, ..., n+1\}$.

Definition 2.2.2. [178] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\mathcal{T}$ the set of all distribution functions defined on $(-\infty, \infty)$ and $\varepsilon_0(t)$ denote the specific distribution function, *i.e.*,

$$\varepsilon_0(t) = \begin{cases} 1, & \text{if } t > 0; \\ 0, & \text{if } t \leq 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \mathcal{T}$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.2.3. [178] A Menger PM-space is a triplet $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a PM-space and Δ is a t-norm satisfying the following condition:

$$F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s)),$$

for all $x, y, z \in X$ and $t, s \geq 0$.

Lemma 2.2.1. [132] Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 2.2.4. [133] A Menger PQM-space is a triplet $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a PM-space and Δ is a continuous t-norm satisfying: for all $x, y, z \in X$ and $t, s > 0$
(i) $F_{x,y}(t) = \varepsilon_0(t)$ and $F_{y,x}(t) = \varepsilon_0(t)$, then $x = y$;
(ii) $F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$.

A Menger PQM-space is called a Menger PM-space if it satisfies the symmetry condition, i.e., $F_{x,y}(t) = F_{y,x}(t)$.

Definition 2.2.5. [131] Let $(X, \mathcal{F}, \Delta)$ be a Menger PQM-space. A sequence $\{x_n\}$ in X is said to be
(i) F-convergent to $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists $k \in \mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, whenever $n \geq k$;
(ii) left Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists $k \in \mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n \geq m \geq k$;

(iii) A Menger PQM-space $(X, \mathcal{F}, \Delta)$ is called left complete if every left Cauchy sequence is F-convergent to a point in X.

Definition 2.2.6. [73] A t-norm Δ is of Hadzic type (in short, H-type) and $\Delta \in \mathcal{H}$ if the family $\{\Delta^n\}_{n\in\mathbb{N}}$ of its iterates defined, for each x in $[0,1]$ by $\Delta^0(x) = 1$, $\Delta^{n+1}(x) = \Delta(\Delta^n(x), x)$, $\forall\ n \geq 0$ is equicontinuous at $x = 1$, *i.e.*, for all $\epsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that $x > 1-\delta \Rightarrow \Delta^n(x) > 1-\epsilon$, $\forall\ n \geq 1$.

There is a nice characterization of continuous t-norm Δ of the class $\mathcal{H}$ in [164].

The t-norm Δ_M is an trivial example of a t-norm of H-type, but there are t-norms Δ of Hadzic type with $\Delta \neq \Delta_M$ (see examples in [73]).

Definition 2.2.7. [73] If Δ is a t-norm and $\{x_1, x_2, ..., x_n\} \in [0,1]^n$ $(n \in \mathbb{N})$, then $\Delta_{i=1}^n x_i$ is defined recurrently by 1, if $n = 0$ and $\Delta_{i=1}^n x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$, for all $n \geq 1$. If $\{x_i\}_{i\in\mathbb{N}}$ is a sequence of numbers from $[0,1]$, then $\Delta_{i=1}^\infty x_i$ is defined as $\lim\limits_{n\to\infty} \Delta_{i=1}^n x_i$ (this limit always exists) and $\Delta_{i=n}^\infty x_i$ as $\Delta_{i=1}^\infty x_{n+i}$.

In probabilistic metric spaces, the t-norms Δ and sequences $\{x_n\} \subset [0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$ and $\lim\limits_{n\to\infty} \Delta_{i=1}^\infty x_{n+i} = 1$, are of particular interest.

Proposition 2.2.1. [73]

(i) If $\Delta \geq \Delta_L$, then the following implication holds:

$$\lim_{n\to\infty} \Delta_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty}(1 - x_n) < \infty.$$

(ii) If $\Delta \in \mathcal{H}$, then for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$, one has $\lim\limits_{n\to\infty} \Delta_{i=1}^\infty x_{n+i} = 1$.

Note that if Δ is a t-norm for which there exists $\{x_n\} \subset [0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$ and $\lim\limits_{n\to\infty} \Delta_{i=1}^\infty x_{n+i} = 1$, then $\sup_{t<1} T(t,t) = 1$.

Proposition 2.2.2. [73] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of numbers from $[0,1]$ such that $\lim\limits_{n\to\infty} x_n = 1$ and the t-norm Δ is of H-type. Then

$$\lim_{n\to\infty} \Delta_{i=n}^\infty x_i = \lim_{n\to\infty} \Delta_{i=1}^\infty x_{n+i} = 1.$$

Lemma 2.2.2. [73] If a Menger PQM-space $(X, \mathcal{F}, \Delta)$ satisfies $F_{x,y}(t) = C$, for all $t > 0$ with fixed $x, y \in X$. Then we have $C = 1$ and $x = y$.

Lemma 2.2.3. [73] Let the function $\phi(t)$ satisfy the condition (Φ) : $\phi(t) : [0,\infty) \to [0,\infty)$ is non-decreasing and $\sum\limits_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, where $\phi^n(t)$ denotes the n^{th} iterative function of $\phi(t)$. Then $\phi(t) < t$ for all $t > 0$.

Definition 2.2.8. [132] A pair (A, S) of self mappings defined on a Menger PQM-space $(X, \mathcal{F}, \Delta)$ is said to be compatible if $F_{ASx_n, SAx_n}(t) \to \varepsilon_0(t)$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to x$ for some $x \in X$ as $n \to \infty$.

Definition 2.2.9. [156] A pair (A, S) of self-mappings defined on a Menger PQM-space $(X, \mathcal{F}, \Delta)$ is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Remark 2.2.1. [187] If two self-mappings of a Menger PQM-space $(X, \mathcal{F}, \Delta)$ are compatible then they are weakly compatible but the converse is not true.

Definition 2.2.10. [124] A pair (A, S) of self-mappings defined on a non-empty set X is called conversely commuting if, for all $x \in X$, $ASx = SAx$ implies $Ax = Sx$.

Definition 2.2.11. [124] Let A and S be self mappings of a non-empty set X. A point $x \in X$ is called commuting point of A and S if $ASx = SAx$.

In 2008, Rezaiyan et al. [171] proved the following common fixed point theorem as a probabilistic generalization of Banach's contraction principle.

Theorem 2.2.1. ([171], Theorem 2.2) *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space and let $f, g : X \to X$ be two self-mappings satisfying:*
(i) $g(X) \subseteq f(X)$;
(ii) f is continuous;
(iii) $F_{g(x),g(y)}(kt) \geq F_{f(x),f(y)}(t)$, for all $x, y \in X$ and for some $0 < k < 1$,
then f and g have a unique common fixed point provided f and g commute.

In 2009, Mihet [131] obtained the result of Rezaiyan et al. [171] is invalid and proved a common fixed point theorem in Menger PQM-space under two necessary additional conditions. Some examples are also mentioned in [131] which support the necessity of these additional conditions.

Theorem 2.2.2. ([131], Theorem 2.1) *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space and let $f, g : X \to X$ be two self-mappings satisfying:*
(i) Δ is of Hadzic type;
(ii) every convergent sequence in X has a unique limit;
(iii) $g(X) \subseteq f(X)$;
(iv) f is continuous;
(v) $F_{g(x),g(y)}(kt) \geq F_{f(x),f(y)}(t)$, for all $x, y \in X$ and for some $0 < k < 1$,
then f and g have a unique common fixed point provided f and g commute.

Pant and Chauhan [156] improved the result of Mihet [131] by taking closedness of one of the underlying subspaces and weakly compatible mappings which is more general than commutativity.

Theorem 2.2.3. ([156], Theorem 2.1) *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space and let $A, B, L : X \to X$ be two self mappings satisfying:*
(i) Δ is of Hadzic type;
(ii) every convergent sequence in X has a unique limit;
(iii) $AB(X) \subseteq L(X)$;
(iv) $L(X)$ is a closed subset of X;
(v) There exists a constant $k \in (0,1)$ such that

$$F_{AB(x),AB(y)}(kt) \geq F_{L(x),L(y)}(t),$$

for all $x, y \in X$ and $t > 0$,
then A, B and L have a unique common fixed point provided the pair (L, AB) is weakly compatible.

2.3 Implicit Functions

In 1999, Popa [161] initiated the idea of implicit functions rather than a single contraction condition to prove fixed point theorems in metric spaces whose strength lies in its unifying power as an implicit function can cover several contraction conditions in one go which include known as well as unknown contraction conditions. This fact is evident from examples furnished in Popa [161]. In 2005, Singh and Jain [189] studied the following class of implicit functions and obtained some fixed point results in framework of fuzzy metric spaces.

Let Φ be the set of all real continuous functions $\phi : [0,1]^4 \to \mathbb{R}$, non-decreasing in first argument and satisfying the following conditions:

(ϕ_1) For $u, v \geq 0$, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies that $u \geq v$,

(ϕ_2) $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 2.3.1. Define $\phi : [0,1]^4 \to \mathbb{R}$, as $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

In 2008, Imdad and Ali [89] used the following class of implicit functions for the existence of a common fixed point in fuzzy metric spaces.

Let Ψ denote the family of all continuous functions $\psi : [0,1]^4 \to \mathbb{R}$, satisfying the following conditions:

(ψ_1) For every $u > 0$, $v \geq 0$, with $\psi(u,v,u,v) \geq 0$ or $\psi(u,v,v,u) \geq 0$ we have $u > v$,

(ψ_2) $\psi(u,u,1,1) < 0$ implies that $u > 0$.

Imdad and Ali [89] also contains the following examples of class of implicit functions Ψ.

Example 2.3.2. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1 - \varphi(\min\{t_2,t_3,t_4\})$, where $\varphi : [0,1] \to [0,1]$ is a continuous function such that $\varphi(s) > s$ for $0 < s < 1$.

Example 2.3.3. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1 - k \min\{t_2,t_3,t_4\}$, where $k > 1$.

Example 2.3.4. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1 - kt_2 - \min\{t_3,t_4\}$, where $k > 1$.

Example 2.3.5. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1 - at_2 - bt_3 - ct_4$, where $a > 1$ and $b, c \geq 0$ $(b, c \neq 1)$.

Example 2.3.6. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1 - at_2 - b(t_3 + t_4)$, where $a > 1$ and $0 \leq b < 1$.

Example 2.3.7. Define $\psi : [0,1]^4 \to \mathbb{R}$, as $\psi(t_1,t_2,t_3,t_4) = t_1^3 - kt_2t_3t_4$, where $k > 1$.

In 2011, Gopal *et al.* [69] showed that the above mentioned classes of functions Φ and Ψ are independent classes.

2.4 Results via Conversely Commuting Mappings in Menger PM-spaces

First, we prove a unique common fixed point theorem for two pairs of self-mappings satisfying a class of implicit function Ψ.

Theorem 2.4.1. *Let A, B, S and T be four self-mappings of a Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm and the pairs (A, S) and (B, T) are conversely commuting, respectively, and satisfy the following condition:*

$$\psi\big(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\big) \geq 0, \tag{2.4.1}$$

for all $x, y \in X$, $t > 0$ and $\psi \in \Psi$. If A and S have a commuting point and B and T have a commuting point, then A, B, S and T have a unique common fixed point in X.

Proof. Let u be the commuting point of A and S. Then $ASu = SAu$. And let v be the commuting point of B and T. Then $BTv = TBv$. Since A and S are conversely commuting, we have $Au = Su$. Since B and T are conversely commuting, we have $Bv = Tv$. Hence, $AAu = ASu = SAu = SSu$ and $BBv = BTv = TBv = TTv$.

(i) We claim that $Au = Bv$. On using (2.4.1) with $x = u$, $y = v$, we get

$$\psi\big(F_{Au,Bv}(t), F_{Su,Tv}(t), F_{Au,Su}(t), F_{Bv,Tv}(t)\big) \geq 0,$$

or, equivalently,

$$\psi\big(F_{Au,Bv}(t), F_{Au,Bv}(t), 1, 1\big) \geq 0.$$

Hence, for $F_{Au,Bv}(t) = 1$ for all $t > 0$, we have $Au = Bv$. Thus $Au = Su = Bv = Tv$.

(ii) Now, we show that Au is a fixed point of mapping A. In order to establish this, using (2.4.1) with $x = Au$, $y = v$, we have

$$\psi\big(F_{AAu,Bv}(t), F_{SAu,Tv}(t), F_{AAu,SAu}(t), F_{Bv,Tv}(t)\big) \geq 0,$$

and so

$$\psi\big(F_{AAu,Au}(t), F_{AAu,Au}(t), 1, 1\big) \geq 0.$$

Hence, for $F_{AAu,Au}(t) = 1$ for all $t > 0$, we get $AAu = Au$. Similarly we show that $Bv = BBv$. On using (2.4.1) with $x = u$, $y = Bv$, we have

$$\psi\big(F_{Au,BBv}(t), F_{Su,TBv}(t), F_{Au,Su}(t), F_{BBv,TBv}(t)\big) \geq 0,$$

or, equivalently,

$$\psi\big(F_{Bv,BBv}(t), F_{Bv,BBv}(t), 1, 1\big) \geq 0.$$

Thus, for $F_{Bv,BBv}(t) = 1$ for all $t > 0$, and we get $BBv = Bv$. Since $Au = Bv$, we have $Au = Bv = BBv = BAu$ which shows that Au is a fixed point of the mapping B. On the other hand, $Au = Bv = BBv = TBv = TAu$ and $Au = AAu = ASu = SAu$. Hence, $Au(= w \in X)$ is a common fixed point of A, B, S and T.

(iii) For the uniqueness of common fixed point, we use (2.4.1) with $x = w$ and $y = u'$, where u' is another common fixed point of the mappings A, B, S and T. Now, we have

$$\psi\big(F_{Aw,Bu'}(t), F_{Sw,Tu'}(t), F_{Aw,Sw}(t), F_{Bu',Tu'}(t)\big) \geq 0,$$

and so

$$\psi\big(F_{w,u'}(t), F_{w,u'}(t), 1, 1\big) \geq 0.$$

Hence, we get $w = u'$. Therefore, w is a unique common fixed point of the mappings A, B, S and T.

Corollary 2.4.1. *The conclusions of Theorem 2.4.1 remain true if condition (2.4.1) is replaced by one of the following conditions: for all $x, y \in X$*

$$F_{Ax,By}(t) \geq \varphi\big(\min \{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\}\big), \tag{2.4.2}$$

where $\varphi : [0,1] \to [0,1]$ is a continuous function such that $\varphi(s) > s$ for all $0 < s < 1$;

$$F_{Ax,By}(t) \geq k\big(\min \{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\}\big), \tag{2.4.3}$$

where $k > 1$;

$$F_{Ax,By}(t) \geq k\ F_{Sx,Ty}(t) + \min \{F_{Ax,Sx}(t), F_{By,Ty}(t)\}, \tag{2.4.4}$$

where $k > 0$;

$$F_{Ax,By}(t) \geq a\ F_{Sx,Ty}(t) + b\ F_{Ax,Sx}(t) + c\ F_{By,Ty}(t), \tag{2.4.5}$$

where $a > 1$ and $b, c \geq 0$ $(b, c \neq 1)$;

$$F_{Ax,By}(t) \geq a\ F_{Sx,Ty}(t) + b\ \big[F_{Ax,Sx}(t) + F_{By,Ty}(t)\big], \tag{2.4.6}$$

where $a > 1$ and $0 \leq b < 1$;

$$F_{Ax,By}(t) \geq k\ F_{Sx,Ty}(t)\ F_{Ax,Sx}(t)\ F_{By,Ty}(t), \tag{2.4.7}$$

where $k > 1$.

Proof. The proof of each inequality (2.4.2)-(2.4.7) easily follows from Theorem 2.4.1 in view of Examples 2.3.2-2.3.7.

Now, we state a unique common fixed point theorem satisfying a class of implicit function Φ.

Theorem 2.4.2. *Let A, B, S and T be four self-mappings on a Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm and the pairs (A, S) and (B, T) are conversely commuting, respectively, and satisfying:*

$$\phi\big(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(kt)\big) \geq 0.$$

$$\phi\big(F_{Ax,By}(kt), F_{Sx,Ty}(t), F_{Ax,Sx}(kt), F_{By,Ty}(t)\big) \geq 0,$$

for all $x, y \in X$, $t > 0$, $k \in (0, 1)$ and $\phi \in \Phi$. If A and S have a commuting point and B and T have a commuting point, then A, B, S and T have a unique common fixed point in X.

Proof. The proof of this theorem can be completed on the lines of the proof of Theorem 2.4.1 (in view of Lemma 2.2.1), hence details are omitted.

By choosing A, B, S and T suitably, we can deduce corollaries involving two as well as three self-mappings. For the sake of naturality, we only derive the following corollary (due to Theorem 2.4.1) involving a pair of self-mappings.

Corollary 2.4.2. *Let A and S be two self-mappings of a Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm and the mappings A and S are conversely commuting satisfying*

$$\psi\big(F_{Ax,Ay}(t), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t)\big) \geq 0,$$

for all $x, y \in X$, $t > 0$ and $\psi \in \Psi$. If A and S have a commuting point, then A and S have a unique common fixed point in X.

2.5 Illustrative Examples

In this section, we present some examples illustrating the results proved in Section 2.4.

Example 2.5.1. Let $X = [1, \infty)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Define $\Delta(a, b) = \min\{a, b\}$. Clearly $(X, \mathcal{F}, \Delta)$ is a Menger PM-space. Let $\psi : [0, 1]^4 \to \mathbb{R}$ as $\psi(t_1, t_2, t_3, t_4) = t_1 - \varphi(\min\{t_2, t_3, t_4\})$ with $\varphi(s) = \sqrt{s}$ for $0 < s < 1$. Define the self-mappings A, B, S and T by

$$A(x) = \begin{cases} 2x - 1, & \text{if } x < 2; \\ 1, & \text{if } x \geq 2, \end{cases}$$

$$S(x) = \begin{cases} x^2, & \text{if } x < 2; \\ x + 3, & \text{if } x \geq 2, \end{cases}$$

$$B(x) = \begin{cases} 2x - 1, & \text{if } x < 2; \\ 2, & \text{if } x \geq 2, \end{cases}$$

$$T(x) = \begin{cases} 3x^2 - 2, & \text{if } x < 2; \\ x^2 + 1, & \text{if } x \geq 2. \end{cases}$$

Then the pairs (A, S) and (B, T) are conversely commuting and 1 is a unique common fixed point of the mappings A, B, S and T.

Example 2.5.2. In the setting of Example 2.5.1, define $\phi : [0, 1]^4 \to \mathbb{R}$ as $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 - 8t_3 + 10t_4$ besides retaining the rest. Then all the conditions of Theorem 2.4.2 are satisfied for some fixed $k \in (0, 1)$ and 1 is a unique common fixed point of the mappings A, B, S and T.

2.6 Results via Weakly Compatible Mappings in Men PQM-Spaces

In this section, we prove a common fixed point theorem for finite number of self mappings in Menger PQM-spaces using weak compatibility.

Theorem 2.6.1. *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space and $f_1, f_2, ..., f_n$ and g be the finite number of self-mappings satisfying:*
(i) Δ is of Hadzic type;
(ii) every convergent sequence in X has a unique limit;
(iii) $g(X) \subset f_1 f_2 ... f_n(X)$;
(iv) for all $x, y \in X$ and $t > 0$,

$$F_{g(x),g(y)}\big(\phi(t)\big) \geq F_{f_1 f_2 ... f_n(x), f_1 f_2 ... f_n(y)}(t), \tag{2.6.1}$$

where the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ);
(v) one of $g(X)$ and $f_1 f_2 ... f_n(X)$ is a complete subspace of X, then $f_1 f_2 ... f_n$ and g have

a coincidence point.
(vi) Suppose that

$$
\begin{aligned}
g(f_2 \ldots f_n) &= (f_2 \ldots f_n)g, \\
g(f_3 \ldots f_n) &= (f_3 \ldots f_n)g, \\
&\vdots \\
g f_n &= f_n g, \\
f_1(f_2 \ldots f_n) &= (f_2 \ldots f_n)f_1, \\
f_1 f_2(f_3 \ldots f_n) &= (f_3 \ldots f_n)f_1 f_2, \\
&\vdots \\
f_1 \ldots f_{n-1}(f_n) &= (f_n)f_1 \ldots f_{n-1}.
\end{aligned}
$$

Moreover, the mappings $f_1, f_2, \ldots, f_n$ and g have a unique common fixed point in X provided that the pair $(f_1 f_2 \ldots f_n, g)$ is weakly compatible.

Proof. Let x_0 be an arbitrary element in X. By (iii), we can find x_1 such that $f_1 f_2 \ldots f_n(x_1) = g(x_0)$. By induction, we can find a sequence $\{x_n\} \in X$ such that $f_1 f_2 \ldots f_n(x_n) = g(x_{n-1})$.

$$
\begin{aligned}
F_{f_1 f_2 \cdots f_n(x_n), f_1 f_2 \cdots f_n(x_{n-1})}(t) &= F_{g(x_{n-1}), g(x_{n-2})}(t) \\
&\geq F_{f_1 f_2 \cdots f_n(x_{n-1}), f_1 f_2 \cdots f_n(x_{n-2})}\big(\phi^{-1}(t)\big) \\
&\geq \ldots \geq F_{f_1 f_2 \cdots f_n(x_0), f_1 f_2 \cdots f_n(x_1)}\big(\phi^{-n}(t)\big),
\end{aligned}
$$

for $n = 1, 2, \ldots$. First we show that $\{y_n\}$, $y_n = f_1 f_2 \ldots f_n(x_n)$ is a left Cauchy sequence. Let $\varepsilon > 0$ be given and $\lambda \in (0,1)$ be such that $T^{m-1}(1-\lambda, \ldots, 1-\lambda) > 1-\varepsilon$. Also assume that for $t > 0$ such that $F_{y_0, y_1}(t) > 1-\lambda$, δ be a positive number and $n_1 \in \mathbb{N}$ be such that $\sum\limits_{i=n_1}^{\infty} \phi^i(t) \leq \delta$. Then, for every $n \geq n_1$ and $m \in \mathbb{N}$, we have

$$
\begin{aligned}
F_{y_n, y_{n+m}}(\delta) &\geq F_{y_n, y_{n+m}}\left(\sum_{i=n}^{n+m-1} \phi^i(t)\right) \\
&\geq T^{m-1}\big(F_{y_n, y_{n+1}}(\phi^n(t)), \ldots, F_{y_{n+m-1}, y_{n+m}}(\phi^{n+m-1}(t))\big) \\
&\geq T^{m-1}(1-\lambda, \ldots, 1-\lambda) \\
&> 1-\varepsilon.
\end{aligned}
$$

Hence, $\{y_n\}$ is a left Cauchy sequence in X. Since the space $(X, \mathcal{F}, \Delta)$ is left complete, there exists $z \in X$ such that $\lim\limits_{n\to\infty} g(x_{n-1}) = \lim\limits_{n\to\infty} f_1 f_2 \ldots f_n(x_n) = z$. Suppose that

$f_1 f_2 ... f_n(X)$ is a complete subspace of X, then the subsequence $\{y_{2n+1}\}$ which is contained in $f_1 f_2 ... f_n(X)$ must get a limit z in $f_1 f_2 ... f_n(X)$, *i.e.*, $f_1 f_2 ... f_n(u) = z$ for some $u \in X$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$ Therefore, the sequence $\{y_n\}$ also converge implying thereby the convergence of another subsequence $\{y_{2n}\}$. Putting $x = u$ and $y = x_{2n+1}$ in (2.6.1), we get

$$F_{g(u),g(x_{2n+1})}\big(\phi(t)\big) \geq F_{f_1 f_2 ... f_n(u), f_1 f_2 ... f_n(x_{2n+1})}(t),$$

$$F_{g(u),z}\big(\phi(t)\big) \geq F_{z,z}(t) = 1.$$

Hence, $F_{g(u),z}\big(\phi(t)\big) = 1$. Thus $g(u) = z$. Since the pair $(g, f_1 f_2 ... f_n)$ is weakly compatible, we have

$$f_1 f_2 ... f_n(gu) = g(f_1 f_2 ... f_n u) = gz.$$

On using (2.6.1) with $x = z$ and $y = x_{2n+1}$, we have

$$F_{g(z),g(x_{2n+1})}\big(\phi(t)\big) \geq F_{f_1 f_2 ... f_n(z), f_1 f_2 ... f_n(x_{2n+1})}(t),$$

$$F_{g(z),z}\big(\phi(t)\big) \geq F_{g(z),z}(t).$$

Since F is non-decreasing, we get $F_{g(z),z}\big(\phi(t)\big) \leq F_{g(z),z}(t)$. Hence $F_{g(z),z}(t) = C$, for all $t > 0$. From Lemma 2.2.2, we conclude that $C = 1$, *i.e.*, $g(z) = z$. Therefore $f_1 f_2 ... f_n(z) = gz = z$. We show that z is the fixed point of all the component mappings Putting $x = f_2 ... f_n z, y = z$ and $f_1' = f_1 f_2 ... f_n$ in (2.6.1), we obtain

$$F_{g(f_2 ... f_n z),g(z)}\big(\phi(t)\big) \geq F_{f_1'(f_2 ... f_n z), f_1' z}(t),$$

$$F_{f_2 ... f_n(z),z}\big(\phi(t)\big) \geq F_{f_2 ... f_n(z),z}(t).$$

On the other hand, since F is non-decreasing, we get $F_{f_2 ... f_n(z),z}\big(\phi(t)\big) \leq F_{f_2 ... f_n(z),z}(t$ Hence, $F_{f_2 ... f_n(z),z}(t) = C$, for all $t > 0$. In view of Lemma 2.2.2, we get $C = 1$. *i.e* $f_2 ... f_n(z) = z$. Thus, $f_1 z = f_1(f_2 ... f_n)z = z$. Similarly, we have $f_2 z = f_3 z = ... = f_n z =$ z. So z is the common fixed point of $f_1, f_2, ..., f_n$ and g.

Uniqueness: Let w $(w \neq z)$ be another common fixed point of $f_1, f_2, ..., f_n$ and g. Puttin $x = z$ and $y = w$ in (2.6.1), we get

$$F_{g(z),g(w)}\big(\phi(t)\big) \geq F_{f_1 f_2 ... f_n(z), f_1 f_2 ... f_n(w)}(t),$$

$$F_{z,w}\big(\phi(t)\big) \geq F_{z,w}(t).$$

Since F is non-decreasing, we have $F_{z,w}\big(\phi(t)\big) \leq F_{z,w}(t)$. Hence $F_{z,w}(t) = C$ for all $t >$ (Appealing to Lemma 2.2.2, we obtain $z = w$ which shows the uniqueness of the commo

fixed point.
The proof is similar when $g(X)$ is assumed to be a complete subspace of X.

Corollary 2.6.1. *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space. Further, let f and g be two self-mappings of X satisfying:*
(i) Δ is of Hadzic type;
(ii) every convergent sequence in X has a unique limit;
(iii) $g(X) \subset f(X)$;
(iv) for all $x, y \in X$ and $t > 0$,

$$F_{g(x),g(y)}(\phi(t)) \geq F_{f(x),f(y)}(t),$$

where the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ);
(v) one of $g(X)$ and $f(X)$ is a complete subspace of X, then f and g have a coincidence point.
Moreover, the mappings f and g have a unique common fixed point in X provided the pair (f, g) is weakly compatible.

Proof. If we set $f_1 f_2 ... f_n = f$ in Theorem 2.6.1, then the result easily follows.

Remark 2.6.1. Theorem 2.6.1 improves and extends the results of Rezaiyan et al. ([171], Theorem 2.2), Mihet et al. ([131], Theorem 2.1), Pant and Chauhan ([156], Theorem 2.1 and Corollary 2.2) and Sastry et al. ([176], Theorem 2.3 and Corollary 2.5).

It should be noticed that (see [129], Theorem 3.3 for the case $gx = x$) the condition Δ is of Hadzic type in Theorem 2.6.1 and Corollary 2.6.1 may be replaced by $\lim\limits_{n\to\infty} T_{i=n}^{\infty} F_{f_1 f_2 ... f_n(x),gx}\left(\frac{1}{\mu^i}\right) = 1$ and $\lim\limits_{n\to\infty} T_{i=n}^{\infty} F_{f(x),g(x)}\left(\frac{1}{\mu^i}\right) = 1$, for some $x \in X$ and some $\mu \in (0, 1)$. Taking into account Proposition 2.2.1, we get the following natural results:

Theorem 2.6.2. *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space. Further, let $f_1, f_2, ..., f_n$ and g be the finite number of self mappings satisfying:*
(i) every convergent sequence in X has a unique limit;
(ii) $g(X) \subset f_1 f_2 ... f_n(X)$;
(iii) for all $x, y \in X$ and $t > 0$,

$$F_{g(x),g(y)}(\phi(t)) \geq F_{f_1 f_2 ... f_n(x), f_1 f_2 ... f_n(y)}(t),$$

where the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ);

(iv) one of $g(X)$ and $f_1f_2...f_n(X)$ is a complete subspace of X, then $f_1f_2...f_n$ and g have a coincidence point;
(v) Suppose that

$$\begin{aligned} g(f_2...f_n) &= (f_2...f_n)g, \\ g(f_3...f_n) &= (f_3...f_n)g, \\ &\vdots \\ gf_n &= f_ng, \\ f_1(f_2...f_n) &= (f_2...f_n)f_1, \\ f_1f_2(f_3...f_n) &= (f_3...f_n)f_1f_2, \\ &\vdots \\ f_1...f_{n-1}(f_n) &= (f_n)f_1...f_{n-1}. \end{aligned}$$

Moreover, the mappings $f_1, f_2, ..., f_n$ and g have a unique common fixed point in X provided that the pair $(f_1f_2...f_n, g)$ is weakly compatible and

$$\sum_{i=1}^{\infty}\left(1 - F_{f_1f_2...f_n(x),g(x)}\left(\frac{1}{\mu^i}\right)\right) < \infty,$$

for some $x \in X$ and $\mu \in (0,1)$.

Theorem 2.6.3. *Let $(X, \mathcal{F}, \Delta)$ be a left complete Menger PQM-space. Further, let f and g be two self mappings satisfying:*
(i) every convergent sequence in X has a unique limit;
(ii) $g(X) \subset f(X)$;
(iii) for all $x, y \in X$ and $t > 0$,

$$F_{g(x),g(y)}(\phi(t)) \geq F_{f(x),f(y)}(t),$$

where the function $\phi(t) : [0,\infty) \to [0,\infty)$ is onto, strictly increasing and satisfies condition (Φ);
(iv) one of $g(X)$ and $f(X)$ is a complete subspace of X, then f and g have a coincidence point.
Moreover, the mappings f and g have a unique common fixed point in X provided that the pair (f, g) is weakly compatible and

$$\sum_{i=1}^{\infty}\left(1 - F_{f(x),g(x)}\left(\frac{1}{\mu^i}\right)\right) < \infty,$$

for some $x \in X$ and $\mu \in (0,1)$.

Remark 2.6.2. Theorem 2.6.2 and Theorem 2.6.3 extend the results of Mihet ([131], Theorem 2.2) and Pant and Chauhan ([156], Corollary 2.3) to finite number of self mappings.

Remark 2.6.3. The conclusion of Theorems 2.6.1-2.6.3 and Corollary 2.6.1 remain true for $\phi(t) = kt$, where $k \in (0,1)$ and $t > 0$.

Chapter 3

Some Even Tupled Coincidence and Fixed Point Theorems in Ordered Metric Spaces

3.1 Introduction

Existence and uniqueness of a fixed point for contraction mappings in ordered metric spaces were discussed first by Ran and Reurings [166] in 2004. This theorem has inspired intense research activity by now there exist an extensive literature on and around this theorem (e.g. [2], [5], [12], [13], [50], [75], [77], [93], [139], [141], [144], [154], [162]).

The notion of coupled fixed point in ordered metric spaces was initiated by Bhaskar and Lakshmikantham [30] in 2006. In this paper they proved some coupled fixed point theorems for mixed monotone mapping under a set of conditions and applied their results to the problems of the existence and uniqueness of solution for the periodic boundary value problems. In 2009, Lakshmikantham and Ćirić [119] extended the results of Bhaskar and Lakshmikantham for nonlinear contractions by defining the notion of coupled coincidence point and mixed g-monotone property. Further, Luong *et al.* [125] proved some coupled fixed point theorems for mappings having the mixed monotone property in ordered complete metric spaces dependent on another function which are generalization of the main results of Bhaskar and Lakshmikantham [30]. Later, various results on coupled fixed point have been obtained, (see [1], [45], [78], [88], [126] ,[140], [174], [175]).

The contents of this chapter are based on two research papers. One of them has been published in Advances in Fixed Point Theory, 3(3), 735-748, (2013) while the other one is communicated in Fixed Point Theory Appl.

In [25], Berinde and Borcut introduced the concept of tripled fixed point and proved some tripled fixed point theorems. After that, Karapinar [98] introduced the concept of a quadruple fixed point and mixed monotone property of a mapping $F : X^4 \to X$ and obtained some quadruple fixed point theorems in ordered complete metric spaces. For the work of this kind one can be referred to ([99]-[101],[137]).

Recently, Imdad *et al.* [91] introduced the concept of even-tupled coincidence point and mixed g-monotone property and utilize these two definitions to obtain even-tupled coincidence point theorems for commuting mappings in ordered complete metric spaces. Dalal *et al.* [53] proved the above mentioned even-tupled coincidence point theorems for compatible mappings.

As usual, this section is devoted to a brief introduction, while Section 3.2 contains preliminaries which include some basic notions and core results. In Section 3.3, we prove existence and uniqueness of some even tupled fixed point theorems for mappings enjoying the mixed monotone property involving an ICS map. In Section 3.4, we prove results on even tupled coincidence and common fixed point for mappings $F : X^n \to X$ and $g : X \to X$ without the assumption of the completeness of the metric space X and without any type of commutativity and compatibility conditions on the mappings F and g. In the last and final section, we furnish an illustrative example to demonstrate the main result proved in Section 3.4. Our results generalize and extend the corresponding results of Luong *et al.* [125], Imdad *et al.* [91], Dalal *et al.* [53] and several results of the existing literature.

3.2 Some Relevant Results

In 2006, Bhaskar and Lakshmikantham [30] proved the following coupled fixed point theorem:

Theorem 3.2.1. [30] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X \times X \to X$ a mapping. Suppose that the following conditions are satisfied:*
(i) F has the mixed monotone property;
(ii) either F is continuous or $(X, d, \preceq)$ has MCB property;
(iii) there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$;
(iv) there exists a constant $k \in [0, 1)$ such that for all $x, y, u, v \in X$,

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)] \quad \text{with} \quad u \preceq x, \ y \preceq v.$$

Then F has a coupled fixed point.

In 2009, Lakshmikantham and Ćirić [119] extended the results of Bhaskar and Lakshmikantham [30] for nonlinear contractions by defining the notion of coupled coincidence point and mixed g-monotone property.

Definition 3.2.1. [119] We denote by Φ the set of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying
(i)$\phi(t) < t$ for each $t > 0$;
(ii) $\lim_{r \to t^+} \phi(r) < t$ for each $t > 0$.

Theorem 3.2.2. [119] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X \times X \to X$, $g : X \to X$ two mappings. Suppose that the following conditions are satisfied:*
(i) $F(X^2) \subseteq g(X)$;
(ii) g is continuous;
(iii) F and g are commutative;
(iv) F has the mixed g-monotone property;
(v) either F is continuous or $(X, d, \preceq)$ has g-MCB property;
(vi) there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$;
(vii) there exists $\phi \in \Phi$ such that

$$d(F(x,y), F(u,v)) \leq \phi\Big(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\Big),$$

for all $x, y, u, v \in X$ for which $g(x) \preceq g(u)$ and $g(v) \preceq g(y)$.
Then F and g have a coupled coincidence point.

In 2011, Luong et al. [125] proved some coupled fixed point theorems for mappings having the mixed monotone property in ordered complete metric spaces dependent on another function which are generalization of the main results of Bhaskar and Lakshmikantham [30].

Definition 3.2.2. ([42, 43, 125, 135]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Theorem 3.2.3. [125] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X \times X \to X$, $T : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) F has the mixed monotone property;
(ii) T is an ICS mapping;
(iii) either F is continuous or $(X, d, \preceq)$ has MCB property;

(iv) there exist $x_0, y_0 \in X$ such that

$$x_0 \preceq F(x_0, y_0) \text{ and } F(y_0, x_0) \preceq y_0;$$

(v) there exists $\phi \in \Phi$ such that

$$d(TF(x,y), TF(u,v)) \leq \frac{1}{2}\phi\big(d(Tx, Tu) + d(Ty, Tv)\big),$$

$\forall\, x, y, u, v \in X$ for which $u \preceq x$ and $y \preceq v$. Then there exist $x, y \in X$ such that

$$x = F(x,y) \text{ and } y = F(y,x),$$

i.e., F has a coupled fixed point.

Throughout this chapter, n stands for a general even natural number. Let us denote by X^n the product space $X \times X \times ... \times X$ of n identical copies of X.

For a product X^n of a partially ordered set $(X, \preceq)$, we define a partial ordering in the following way:

For $U = (x^1, x^2, x^3, ..., x^n),\ V = (y^1, y^2, y^3, ..., y^n) \in X^n$

$$U \lesssim V \Leftrightarrow x^1 \preceq y^1,\ y^2 \preceq x^2, x^3 \preceq y^3, ..., y^n \preceq x^n.$$

We say that U and V are equal (*i.e.*, $U = V$) if

$$x^1 = y^1,\ x^2 = y^2, x^3 = y^3, ..., x^n = y^n.$$

Also we say that U and V are comparable if either $U \lesssim V$ or $V \lesssim U$.

In 2013, Imdad et al. [91] proved the following result:

Theorem 3.2.4. [91] Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:
(i) $F(X^n) \subseteq g(X)$;
(ii) F has the mixed g-monotone property;
(iii) g is continuous and monotonic increasing;
(iv) (g, F) is a commutating pair;
(v) either F is continuous or $(X, d, \preceq)$ has MCB property;
(vi) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{aligned} gx_0^1 &\preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n), \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) &\preceq gx_0^2, \\ gx_0^3 &\preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2), \end{aligned}$$

$$\vdots$$
$$F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq gx_0^n;$$

(vii) there exists $\phi \in \Phi$ such that

$$d\big(F(U), F(V)\big) \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gy^i)\Big),$$

for all $U = (x^1, x^2, x^3, ..., x^n), V = (y^1, y^2, y^3, ..., y^n) \in X^n$, with $gx^1 \preceq gy^1$, $gy^2 \preceq gx^2$, $gx^3 \preceq gy^3$,........, $gy^n \preceq gx^n$. Then F and g have an n-tupled coincidence point.

Definition 3.2.3. [87] Two mappings $F : X^n \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{cases} \lim_{m\to\infty} d\big(g(F(x_m^1, x_m^2, x_m^3, ..., x_m^n)), F(gx_m^1, gx_m^2, gx_m^3, ..., gx_m^n)\big) = 0 \\ \lim_{m\to\infty} d\big(g(F(x_m^2, x_m^3, ..., x_m^n, x_m^1)), F(gx_m^2, gx_m^3, ..., gx_m^n, gx_m^1)\big) = 0 \\ \vdots \\ \lim_{m\to\infty} d\big(g(F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1})), F(gx_m^n, gx_m^1, gx_m^2, ..., gx_m^{n-1})\big) = 0, \end{cases}$$

where $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ are sequences in X such that

$$\begin{cases} \lim_{m\to\infty} F(x_m^1, x_m^2, x_m^3, ..., x_m^n) = \lim_{m\to\infty} gx_m^1 = x^1 \\ \lim_{m\to\infty} F(x_m^2, x_m^3, ..., x_m^n, x_m^1) = \lim_{m\to\infty} gx_m^2 = x^2 \\ \vdots \\ \lim_{m\to\infty} F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}) = \lim_{m\to\infty} gx_m^n = x^n, \end{cases}$$

for some $x^1, x^2, ..., x^n \in X$ are satisfied.

In 2014, Dalal et al. [53] proved the following result:

Theorem 3.2.5. [53] Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:
(i) $F(X^n) \subseteq g(X)$;
(ii) F has the mixed g-monotone property;
(iii) g is continuous and monotonic increasing;
(iv) the pair (g, F) is compatible;
(v) either F is continuous or $(X, d, \preceq)$ has MCB property;
(vi) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n),$$

$$F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq gx_0^2,$$
$$gx_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2),$$
$$\vdots$$
$$F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq gx_0^n;$$

(vii) there exists $\phi \in \Phi$ such that

$$d\big(F(U), F(V)\big) \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gy^i)\Big),$$

for all $U = (x^1, x^2, x^3, ..., x^n), V = (y^1, y^2, y^3, ..., y^n) \in X^n$, with $gx^1 \preceq gy^1$, $gy^2 \preceq gx^2$, $gx^3 \preceq gy^3$,........, $gy^n \preceq gx^n$. Then F and g have an n-tupled coincidence point.

3.3 n-Tupled Fixed Point Theorems Involving an ICS Map

In this section, we prove some n-tupled fixed point theorems for mapping having the mixed monotone property in ordered complete metric spaces involving an ICS map.

Theorem 3.3.1. *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$, $T : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) F has the mixed monotone property;
(ii) T is an ICS mapping;
(iii) either F is continuous or $(X, d, \preceq)$ has MCB property;
(iv) there exist $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n),$$
$$F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq x_0^2,$$
$$x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2),$$
$$\vdots$$
$$F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq x_0^n;$$

(v) there exists $\phi \in \Phi$ such that

$$d\big(TF(U), TF(V)\big) \leq \frac{1}{n}\phi\Big(\sum_{i=1}^{n} d(Tx^i, Ty^i)\Big), \tag{3.3.1}$$

for all $U = (x^1, x^2, x^3, ..., x^n), V = (y^1, y^2, y^3, ..., y^n) \in X^n$, with $y^1 \preceq x^1$, $x^2 \preceq y^2$, $y^3 \preceq x^3$,........, $x^n \preceq y^n$. Then F has an n-tupled fixed point in X.

Proof. Let $x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq x_0^n. \end{cases} \tag{3.3.2}$$

Choose $x_1^1, x_1^2, x_1^3, ..., x_1^n \in X$ such that

$$\begin{cases} x_1^1 = F(x_0^1, x_0^2, x_0^3, ..., x_0^n) \\ x_1^2 = F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \\ x_1^3 = F(x_0^3, ..., x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_1^n = F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}). \end{cases} \tag{3.3.3}$$

Continuing this process, we construct sequences $\{x_m^1\}, \{x_m^2\}, \{x_m^3\}, ..., \{x_m^n\}$ in X as follows:

$$\begin{cases} x_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, ..., x_m^n) \\ x_{m+1}^2 = F(x_m^2, x_m^3, ..., x_m^n, x_m^1) \\ x_{m+1}^3 = F(x_m^3, ..., x_m^n, x_m^1, x_m^2) \\ \vdots \\ x_{m+1}^n = F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}). \end{cases} \tag{3.3.4}$$

We are going to divide the proof into several steps in order to make it easily readable.

Step 1. We shall prove that for all $m \geq 0$,

$$x_m^1 \preceq x_{m+1}^1, \; x_{m+1}^2 \preceq x_m^2, \; x_m^3 \preceq x_{m+1}^3, ..., x_{m+1}^n \preceq x_m^n. \tag{3.3.5}$$

By using (3.3.2) and (3.3.3), we have

$$\begin{aligned} x_0^1 &\preceq F(x_0^1, x_0^2, x_0^3, ..., x_0^n) = x_1^1, \\ x_1^2 &= F(x_0^2, x_0^3, ..., x_0^n, x_0^1) \preceq x_0^2, \\ x_0^3 &\preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2) = x_1^3, \\ &\vdots \\ x_1^n &= F(x_0^n, x_0^1, x_0^2, ..., x_0^{n-1}) \preceq x_0^n. \end{aligned}$$

So, (3.3.5) holds for $m = 0$. Suppose that (3.3.5) holds for some $m > 0$. As F has the mixed monotone property, we have from (3.3.4) that

$$\begin{aligned} x^1_{m+1} = F(x^1_m, x^2_m, x^3_m, ..., x^n_m) &\preceq F(x^1_{m+1}, x^2_m, x^3_m, ..., x^n_m) \\ &\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_m, ..., x^n_m) \\ &\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_m) \\ &\quad \vdots \\ &\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}) \\ &= x^1_{m+2}, \end{aligned}$$

$$\begin{aligned} x^2_{m+2} = F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}) &\preceq F(x^2_m, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}) \\ &\preceq F(x^2_m, x^3_m, ..., x^n_{m+1}, x^1_{m+1}) \\ &\quad \vdots \\ &\preceq F(x^2_m, x^3_m, ..., x^n_m, x^1_{m+1}) \\ &\preceq F(x^2_m, x^3_m, ..., x^n_m, x^1_m) \\ &= x^2_{m+1}. \end{aligned}$$

Also for the same reason,

$$x^3_{m+1} = F(x^3_m, ..., x^n_m, x^1_m, x^2_m) \preceq F(x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}, x^2_{m+1}) = x^3_{m+2},$$

$$\vdots$$

$$x^n_{m+2} = F(x^n_{m+1}, x^1_{m+1}, x^2_{m+1}, ..., x^{n-1}_{m+1}) \preceq F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = x^n_{m+1}.$$

Thus by the mathematical induction, we conclude that (3.3.5) holds for all $m \geq 0$. This completes the proof of our claim.

Assume that for some $m \in \mathbb{N} \cup \{0\}$,

$$x^1_m = x^1_{m+1},\ x^2_m = x^2_{m+1},\ x^3_m = x^3_{m+1}, ..., x^n_m = x^n_{m+1},$$

then by (3.3.4), $(x^1_m, x^2_m, x^3_m, ..., x^n_m)$ is an n-tupled fixed point of F. Therefore, in the rest of the proof, for any $m \in \mathbb{N} \cup \{0\}$ we will assume that at least

$$x^1_m \neq x^1_{m+1} \text{ or } x^2_m \neq x^2_{m+1} \text{ or } x^3_m \neq x^3_{m+1} \text{ or } ...... \text{ or } x^n_m \neq x^n_{m+1}.$$

Step 2. We shall show that

$$\lim_{m\to\infty} \left(d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1})\right) = 0.$$

Due to (3.3.1) and (3.3.4), we have

$$d(Tx_m^1, Tx_{m+1}^1)$$

$$= d\big(TF(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, ..., x_{m-1}^n), TF(x_m^1, x_m^2, x_m^3, ..., x_m^n)\big),$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx_{m-1}^1, Tx_m^1) + d(Tx_{m-1}^2, Tx_m^2) + d(Tx_{m-1}^3, Tx_m^3) + ... + d(Tx_{m-1}^n, Tx_m^n)\big).$$

Similarly, we can inductively write

$$d(Tx_m^2, Tx_{m+1}^2)$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx_{m-1}^2, Tx_m^2) + d(Tx_{m-1}^3, Tx_m^3) + ... + d(Tx_{m-1}^n, Tx_m^n) + d(Tx_{m-1}^1, Tx_m^1)\big),$$

$$\vdots$$

$$d(Tx_m^n, Tx_{m+1}^n)$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx_{m-1}^n, Tx_m^n) + d(Tx_{m-1}^1, Tx_m^1) + d(Tx_{m-1}^2, Tx_m^2) + ... + d(Tx_{m-1}^{n-1}, Tx_m^{n-1})\big).$$

Adding the above inequalities, we obtain

$$d(Tx_m^1, Tx_{m+1}^1) + d(Tx_m^2, Tx_{m+1}^2) + ... + d(Tx_m^n, Tx_{m+1}^n)$$

$$\leq \phi\big(d(Tx_{m-1}^n, Tx_m^n) + d(Tx_{m-1}^1, Tx_m^1) + d(Tx_{m-1}^2, Tx_m^2) + ... + d(Tx_{m-1}^{n-1}, Tx_m^{n-1})\big). \quad (3.3.6)$$

Set

$$d_m = d(Tx_m^1, Tx_{m+1}^1) + d(Tx_m^2, Tx_{m+1}^2) + ... + d(Tx_m^n, Tx_{m+1}^n). \quad (3.3.7)$$

Using (3.3.6) we have,

$$d_m \leq \phi(d_{m-1}). \quad (3.3.8)$$

Since $\phi(t) < t$ for all $t > 0$, it follows from (3.3.8) that $\{d_m\}$ is a decreasing sequence of positive real numbers. Therefore, there exists some $d \geq 0$ such that

$$\lim_{m\to\infty} \big(d(Tx_m^1, Tx_{m+1}^1) + d(Tx_m^2, Tx_{m+1}^2) + ... + d(Tx_m^n, Tx_{m+1}^n)\big) = \lim_{m\to\infty} d_m = d.$$

Assume that $d > 0$, taking $m \to \infty$ in both sides of (3.3.8) and using the property of ϕ, we have

$$d = \lim_{m\to\infty} d_m \leq \lim_{m\to\infty} \phi(d_{m-1}) = \lim_{d_{m-1}\to d+} \phi(d_{m-1}) < d,$$

which is a contradiction. Thus $d = 0$, *i.e.*,

$$\lim_{m\to\infty}\left(d(Tx^1_m, Tx^1_{m+1}) + d(Tx^2_m, Tx^2_{m+1}) + ... + d(Tx^n_m, Tx^n_{m+1})\right) = \lim_{m\to\infty} d_m = 0. \quad (3.3.9)$$

This proves our claim.

Step 3. We shall show that $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ are Cauchy sequences.

Assume on contrary that atleast one of $\{Tx^1_m\}, \{Tx^2_m\}, ..., \{Tx^n_m\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{Tx^1_{m(k)}\}, \{Tx^1_{t(k)}\}$ of $\{Tx^1_m\}$, $\{Tx^2_{m(k)}\}, \{Tx^2_{t(k)}\}$ of $\{Tx^2_m\}$,...,$\{Tx^n_{m(k)}\}, \{Tx^n_{t(k)}\}$ of $\{Tx^n_m\}$ with $m(k) > t(k) \geq k$ such that

$$d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) \geq \varepsilon. \quad (3.3.10)$$

Further, corresponding to $t(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > t(k) \geq k$ satisfying (3.3.10). Then

$$d(Tx^1_{m(k)-1}, Tx^1_{t(k)}) + d(Tx^2_{m(k)-1}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)-1}, Tx^n_{t(k)}) < \varepsilon. \quad (3.3.11)$$

Using (3.3.10),(3.3.11) and the triangular inequality, we have

$$\begin{aligned} r_k &= d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) \\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)-1}) + d(Tx^1_{m(k)-1}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{m(k)-1}) \\ &\quad + d(Tx^2_{m(k)-1}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{m(k)-1}) + d(Tx^n_{m(k)-1}, Tx^n_{t(k)}) \\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)-1}) + d(Tx^2_{m(k)}, Tx^2_{m(k)-1}) + ... + d(Tx^n_{m(k)}, Tx^n_{m(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in above inequality and using (3.3.9), we have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}[d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})] = \varepsilon. \quad (3.3.12)$$

By the triangular inequality

$$\begin{aligned} r_k &= d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) \\ &\leq d(Tx^1_{m(k)}, Tx^1_{m(k)+1}) + d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1}) + d(Tx^1_{t(k)+1}, Tx^1_{t(k)}) \end{aligned}$$

T-9602 Acc. No. .... Aligarh Muslim University

$$+d(Tx^2_{m(k)}, Tx^2_{m(k)+1}) + d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1}) + d(Tx^2_{t(k)+1}, Tx^2_{t(k)})$$

$$+ ... + d(Tx^n_{m(k)}, Tx^n_{m(k)+1}) + d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1}) + d(Tx^n_{t(k)+1}, Tx^n_{t(k)})$$

$$= d_{m(k)} + d_{t(k)} + d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1}) + d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1})$$

$$+ ... + d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1}). \tag{3.3.13}$$

Since $m(k) > t(k)$, we have

$$x^1_{t(k)} \preceq x^1_{m(k)},\ x^2_{m(k)} \preceq x^2_{t(k)},\ x^3_{t(k)} \preceq x^3_{m(k)}, ...,\ x^n_{m(k)} \preceq x^n_{t(k)}.$$

From (3.3.1) and (3.3.4), we have

$$d(Tx^1_{m(k)+1}, Tx^1_{t(k)+1})$$

$$= d\big(TF(x^1_{m(k)}, x^2_{m(k)}, x^3_{m(k)}, ..., x^n_{m(k)}), TF(x^1_{t(k)}, x^2_{t(k)}, x^3_{t(k)}, ..., x^n_{t(k)})\big)$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})\big).$$

Similarly, we can write

$$d(Tx^2_{m(k)+1}, Tx^2_{t(k)+1})$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx^2_{m(k)}, Tx^2_{t(k)}) + d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)}) + d(Tx^1_{m(k)}, Tx^1_{t(k)})\big).$$

$$\vdots$$

$$d(Tx^n_{m(k)+1}, Tx^n_{t(k)+1})$$

$$\leq \tfrac{1}{n}\phi\big(d(Tx^n_{m(k)}, Tx^n_{t(k)}) + d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)}) + ... + d(Tx^{n-1}_{m(k)}, Tx^{n-1}_{t(k)})\big)$$

From the above inequalities, we have

$$r_k \leq d_{m(k)} + d_{t(k)} + \phi\big(d(Tx^1_{m(k)}, Tx^1_{t(k)}) + d(Tx^2_{m(k)}, Tx^2_{t(k)})$$

$$+d(Tx^3_{m(k)}, Tx^3_{t(k)}) + ... + d(Tx^n_{m(k)}, Tx^n_{t(k)})\big)$$

$$= d_{m(k)} + d_{t(k)} + \phi(r_k).$$

Taking $k \to \infty$ in the above inequality, using (3.3.9), (3.3.13) and the property of ϕ, we have

$$\varepsilon = \lim_{k\to\infty} r_k \leq \lim_{k\to\infty} (d_{m(k)} + d_{t(k)} + \phi(r_k)) = \lim_{r_k\to\varepsilon+} \phi(r_k) < \varepsilon,$$

which is a contradiction. Therefore, $\{Tx_m^1\}, \{Tx_m^2\}, ..., \{Tx_m^n\}$ are Cauchy sequences in X. This completes the proof of our claim.

Since X is a complete metric space, $\{Tx_m^1\}, \{Tx_m^2\}, ..., \{Tx_m^n\}$ are convergent sequences. Since T is an ICS mapping, there exist $x^1, x^2, ..., x^n \in X$ such that

$$\lim_{m\to\infty} x_m^1 = x^1, \lim_{m\to\infty} x_m^2 = x^2, ..., \lim_{m\to\infty} x_m^n = x^n. \tag{3.3.14}$$

Since T is continuous, we have

$$\lim_{m\to\infty} Tx_m^1 = Tx^1, \lim_{m\to\infty} Tx_m^2 = Tx^2, ..., \lim_{m\to\infty} Tx_m^n = Tx^n. \tag{3.3.15}$$

Suppose now that F is continuous. By (3.3.4), (3.3.14) and the continuity of F, we obtain

$$\begin{aligned} x^1 = \lim_{m\to\infty} x_{m+1}^1 = \lim_{m\to\infty} F(x_m^1, x_m^2, ..., x_m^n) &= F(\lim_{m\to\infty} x_m^1, \lim_{m\to\infty} x_m^2, ..., \lim_{m\to\infty} x_m^n) \\ &= F(x^1, x^2, ..., x^n), \end{aligned}$$

$$\begin{aligned} x^2 = \lim_{m\to\infty} x_{m+1}^2 = \lim_{m\to\infty} F(x_m^2, x_m^3, ..., x_m^1) &= F(\lim_{m\to\infty} x_m^2, \lim_{m\to\infty} x_m^3, ..., \lim_{m\to\infty} x_m^1,) \\ &= F(x^2, ..., x^n, x^1), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} x^n = \lim_{m\to\infty} x_{m+1}^n = \lim_{m\to\infty} F(x_m^n, x_m^1, ..., x_m^{n-1}) &= F(\lim_{m\to\infty} x_m^n, \lim_{m\to\infty} x_m^1, ..., \lim_{m\to\infty} x_m^{n-1}) \\ &= F(x^n, x^1, ..., x^{n-1}). \end{aligned}$$

Thus, $(x^1, x^2, ..., x^n) \in X^n$ is an n-tupled fixed point of F.

Next assume that $(X, d, \preceq)$ has MCB property. Since $\{x_m^i\}$ is non-decreasing or non-increasing according as i is odd or even and $x_m^i \longrightarrow x^i$ as $m \to \infty$. we have

$$x_m^1 \preceq x^1, \; x^2 \preceq x_m^2, \; x_m^3 \preceq x^3, ..., \; x^n \preceq x_m^n.$$

Consider now,

$$\begin{aligned}
& d\big(Tx^1, TF(x^1,x^2,...,x^n)\big) \\
& \leq d\big(Tx^1, Tx^1_{m+1}\big) + d\big(Tx^1_{m+1}, TF(x^1,x^2,...,x^n)\big) \\
& = d(Tx^1, Tx^1_{m+1}) + d\big(TF(x^1_m, x^2_m, ..., x^n_m), TF(x^1,x^2,...,x^n)\big) \\
& \leq d(Tx^1, Tx^1_{m+1}) + \frac{1}{n}\phi\big(d(Tx^1_m, Tx^1) + d(Tx^2_m, Tx^2) + ... + d(Tx^n_m, Tx^n)\big). \quad (3.3.16)
\end{aligned}$$

Taking $m \to \infty$, (3.3.16) yields that $d(Tx^1, TF(x^1,x^2,...,x^n)) \leq 0$. Hence

$$d\big(Tx^1, TF(x^1,x^2,...,x^n)\big) = 0.$$

Thus, $Tx^1 = TF(x^1,x^2,...,x^n)$ and since T is injective, we get

$$x^1 = F(x^1,x^2,...,x^n).$$

Analogously, we can show that

$$x^2 = F(x^2,x^3,...,x^1),...,x^n = F(x^n,x^1,...,x^{n-1}).$$

Thus, we proved that F has an n-tupled fixed point.
This completes the proof.

Corollary 3.3.1. *Let $(X,d,\preceq)$ be an ordered complete metric space and $F : X^n \to X$ a mapping. Suppose that the following conditions hold:*
(i) F has the mixed monotone property;
(ii) either F is continuous or $(X,d,\preceq)$ has MCB property;
(iii) there exist $x^1_0, x^2_0, x^3_0, ..., x^n_0 \in X$ such that

$$\begin{gathered}
x^1_0 \preceq F(x^1_0, x^2_0, x^3_0, ..., x^n_0) \\
F(x^2_0, x^3_0, ..., x^n_0, x^1_0) \preceq x^2_0 \\
x^3_0 \preceq F(x^3_0, ..., x^n_0, x^1_0, x^2_0) \\
\vdots \\
F(x^n_0, x^1_0, x^2_0, ..., x^{n-1}_0) \preceq x^n_0;
\end{gathered}$$

(iv) there exists $\phi \in \Phi$ such that

$$d(F(U),F(V)) \leq \frac{1}{n}\phi\Big(\sum_{i=1}^{n} d(x^i,y^i)\Big),$$

for all $U = (x^1,x^2,x^3,.....,x^n), V = (y^1,y^2,y^3,.....,y^n) \in X^n$, with $y^1 \preceq x^1$, $x^2 \preceq y^2$, $y^3 \preceq x^3$,........, $x^n \preceq y^n$. Then F has an n-tupled fixed point in X.

Proof. It follows by taking $Tx = x$, for all $x \in X$, in Theorem 3.3.1.

Corollary 3.3.2. *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$, $T : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) F has the mixed monotone property;
(ii) T is an ICS mapping;
(iii) either F is continuous or $(X, d, \preceq)$ has MCB property;
(iv) there exist $x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n \in X$ such that

$$\begin{gathered} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n) \\ F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \preceq F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \ldots\ldots\ldots, x_0^{n-1}) \preceq x_0^n; \end{gathered}$$

(v) there exists $\phi \in \Phi$ such that

$$d(TF(U), TF(V)) \leq \frac{k}{n}\phi\Big(\sum_{i=1}^{n} d(Tx^i, Ty^i)\Big),$$

for all $U = (x^1, x^2, x^3, \ldots\ldots, x^n), V = (y^1, y^2, y^3, \ldots\ldots, y^n) \in X^n$, with $y^1 \preceq x^1$, $x^2 \preceq y^2$, $y^3 \preceq x^3, \ldots\ldots\ldots, x^n \preceq y^n$. Then F has an n-tupled fixed point in X.

Proof. It follows by taking $\phi(s) = ks$, for all $s \in [0, \infty)$, in Theorem 3.3.1.

Remark 3.3.1. Taking $n = 2$ in Theorem 3.3.1 and in Corollaries 3.3.1-3.3.2, we get Theorem 2.1 and Corollaries 2.2-2.3 of Luong et al. [125].

Now, we shall prove the uniqueness of an n-tupled fixed point.

Theorem 3.3.2. *In addition to the hypotheses of Theorem 3.3.1, suppose that for every $(x^1, x^2, x^3, \ldots, x^n)$, $(y^1, y^2, y^3, \ldots, y^n) \in X^n$, there exists $(z^1, z^2, z^3, \ldots, z^n) \in X^n$ which is comparable to $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$. Then F has a unique n-tupled fixed point $(x^1, x^2, x^3, \ldots, x^n)$.*

Proof. The set of n-tupled fixed points of F is non-empty due to Theorem 3.3.1. Assume now, $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ are two n-tupled fixed points of F, *i.e.*,

$$\begin{cases} x^1 = F(x^1, x^2, x^3, \ldots, x^n), \ y^1 = F(y^1, y^2, y^3, \ldots, y^n), \\ x^2 = F(x^2, x^3, \ldots, x^n, x^1), \ y^2 = F(y^2, y^3, \ldots, y^n, y^1), \\ \qquad \vdots \\ x^n = F(x^n, x^1, x^2, \ldots, x^{n-1}), \ y^n = F(y^n, y^1, y^2, \ldots, y^{n-1}). \end{cases} \tag{3.3.17}$$

We shall show that $(x^1, x^2, x^3, ..., x^n)$ and $(y^1, y^2, y^3, ..., y^n)$ are equal. By assumption, there exists $(z^1, z^2, z^3 ..., z^n) \in X^n$ which is comparable to $(x^1, x^2, x^3, ..., x^n)$, and $(y^1, y^2, y^3, ..., y^n)$.

Define sequences $\{z_m^1\}, \{z_m^2\}, ..., \{z_m^n\}$ as follows:

$$z_0^1 = z^1, z_0^2 = z^2, ..., z_0^n = z^n,$$

$$\begin{cases} z_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, ..., z_m^n) \\ z_{m+1}^2 = F(z_m^2, z_m^3, ..., z_m^n, z_m^1) \\ \vdots \\ z_{m+1}^n = F(z_m^n, z_m^1, z_m^3, ..., z_m^{n-1}), \text{ for all } m \geq 0. \end{cases} \tag{3.3.18}$$

Since $(z^1, z^2, z^3, ..., z^n)$ is comparable with $(x^1, x^2, x^3, ..., x^n)$, we may assume that

$$(z_0^1, z_0^2, z_0^3, ..., z_0^n) = (z^1, z^2, z^3, ..., z^n) \lesssim (x^1, x^2, x^3, ..., x^n).$$

Now, we shall prove that

$$(z_m^1, z_m^2, z_m^3, ..., z_m^n) \lesssim (x^1, x^2, x^3, ..., x^n), \text{ for all } m. \tag{3.3.19}$$

Suppose that (3.3.19) holds for some $m > 0$. Then by the mixed monotone property of F, we have

$$\begin{cases} z_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, ..., z_m^n) \preceq F(x^1, x^2, x^3, ..., x^n) = x^1, \\ x^2 = F(x^2, x^3, ..., x^n, x^1) \preceq F(z_m^2, z_m^3, ..., z_m^n, z_m^1) = z_{m+1}^2, \\ \vdots \\ x^n = F(x^n, x^1, x^2, ..., x^{n-1}) \preceq F(z_m^n, z_m^1, z_m^2, ..., z_m^{n-1}) = z_{m+1}^n. \end{cases}$$

Therefore, $(z_{m+1}^1, z_{m+1}^2, z_{m+1}^3, ..., z_{m+1}^n) \lesssim (x^1, x^2, x^3, ..., x^n)$ for all m. Hence (3.3.19) holds.

From (3.3.17), (3.3.18) and (3.3.1), we have

$$\begin{aligned} d(Tx^1, Tz_m^1) &= d\big(TF(x^1, x^2, ..., x^n), TF(z_{m-1}^1, z_{m-1}^2, ..., z_{m-1}^n)\big) \\ &\leq \tfrac{1}{n}\phi\big(d(Tx^1, Tz_{m-1}^1) + d(Tx^2, Tz_{m-1}^2) + ... + d(Tx^n, Tz_{m-1}^n)\big), \end{aligned}$$

$$\begin{aligned} d(Tx^2, Tz_m^2) &= d\big(TF(x^2, ..., x^n, x^1), TF(z_{m-1}^2, ..., z_{m-1}^n, z_{m-1}^1)\big) \\ &\leq \tfrac{1}{n}\phi\big(d(Tx^2, Tz_{m-1}^2) + ... + (Tx^n, Tz_{m-1}^n) + d(Tx^1, Tz_{m-1}^1)\big), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d(Tx^n, Tz_m^n) &= d\big(TF(x^n, x^1, x^2, ..., x^{n-1}), TF(z_{m-1}^n, z_{m-1}^1, z_{m-1}^2 ..., z_{m-1}^{n-1})\big) \\ &\leq \tfrac{1}{n}\phi\big(d(Tx^n, Tz_{m-1}^n) + d(Tx^1, Tz_{m-1}^1) + ... + d(Tx^{n-1}, Tz_{m-1}^{n-1})\big). \end{aligned}$$

Adding the above inequalities we obtain

$$\begin{aligned} &d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n) \\ &\quad \leq \phi\big(d(Tx^1, Tz_{m-1}^1) + d(Tx^2, Tz_{m-1}^2) + ... + d(Tx^n, Tz_{m-1}^n)\big). \end{aligned} \tag{3.3.20}$$

Set

$$\delta_m = d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n).$$

It follows from (3.3.20) and the property of ϕ that $\{\delta_m\}$ is a monotone decreasing sequence of positive real numbers. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{m\to\infty} \delta_m = \lim_{m\to\infty} \big(d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n)\big) = \delta.$$

Assume that $\delta > 0$, taking $m \to \infty$ in both sides of (3.3.20), we have

$$\begin{aligned} \delta &= \lim_{m\to\infty} \big(d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n)\big) \\ &\leq \lim_{m\to\infty} \phi\big(d(Tx^1, Tz_{m-1}^1) + d(Tx^2, Tz_{m-1}^2) + ... + d(Tx^n, Tz_{m-1}^n)\big) \\ &= \lim_{\delta_{m-1}\to\delta+} \phi(\delta_{m-1}) < \delta, \end{aligned}$$

which is a contradiction. Thus, $\delta = 0$, *i.e.*,

$$\lim_{m\to\infty} \big(d(Tx^1, Tz_m^1) + d(Tx^2, Tz_m^2) + ... + d(Tx^n, Tz_m^n)\big) = 0.$$

This yields that

$$\lim_{m\to\infty} d(Tx^1, Tz_m^1) = 0, \ \lim_{m\to\infty} d(Tx^2, Tz_m^2) = 0, ..., \lim_{m\to\infty} d(Tx^n, Tz_m^n) = 0. \tag{3.3.21}$$

Analogously, we can show that

$$\lim_{m\to\infty} d(Ty^1, Tz_m^1) = 0, \ \lim_{m\to\infty} d(Ty^2, Tz_m^2) = 0, ..., \lim_{m\to\infty} d(Ty^n, Tz_m^n) = 0. \tag{3.3.22}$$

Combining (3.3.21) and (3.3.22) yields that $(Tx^1, Tx^2, ..., Tx^n)$ and $(Ty^1, Ty^2, ..., Ty^n)$ are equal. The fact that T is injective gives us $x^1 = y^1, x^2 = y^2, ..., x^n = y^n$.

3.4 n-Tupled Coincidence Point Theorems Without Completeness and Compatibility

We begin this section by defining the following notion:

Definition 3.4.1. Let X be a non-empty set. Mappings $F : X^n \to X$ and $g : X \to X$ are called w-compatible if

$$F(gx^1, gx^2, ..., gx^n) = g\big(F(x^1, x^2, ..., x^n)\big),$$

whenever

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = gx^1 \\ F(x^2, x^3, ..., x^n, x^1) = gx^2 \\ F(x^3, ..., x^n, x^1, x^2) = gx^3 \\ \qquad\vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = gx^n, \end{cases}$$

for all $x^1, x^2, x^3, ..., x^n \in X$.

The following two known results are helpful in order to prove results of this section:

Lemma 3.4.1. [74] Let X be a non-empty set and g a self mapping on X. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.

Lemma 3.4.2. [191] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$, $g : X \to X$ two mappings. If F has mixed g-monotone property and $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$ such that $gx^1 = gy^1, gx^2 = gy^2, ..., gx^n = gy^n$, then

$$\begin{aligned} F(x^1, x^2, x^3, ..., x^n) &= F(y^1, y^2, y^3, ..., y^n) \\ F(x^2, x^3, ..., x^n, x^1) &= F(y^2, y^3, ..., y^n, y^1) \\ F(x^3, ..., x^n, x^1, x^2) &= F(y^3, ..., y^n, y^1, y^2) \\ &\vdots \\ F(x^n, x^1, x^2, x^3, ..., x^{n-1}) &= F(y^n, y^1, y^2, y^3, ..., y^{n-1}). \end{aligned}$$

Now, we present a variant of the main results of Imdad et al. [91] and Dalal et al. [53] without completeness of metric space and without commutativity and compatibility of the mappings involved.

Theorem 3.4.1. *Let $(X, d, \preceq)$ be an ordered metric space and $F : X^n \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) One of $g(X)$ and $F(X^n)$ is complete;
(ii) $F(X^n) \subseteq g(X)$;
(iii) F has the mixed g-monotone property;
(iv) either F and g are continuous or $(gX, d, \preceq)$ has MCB property;
(v) there exist $x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n \in X$ such that

$$\begin{gathered} gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots\ldots, x_0^n) \\ F(x_0^2, x_0^3, \ldots\ldots, x_0^n, x_0^1) \preceq gx_0^2 \\ gx_0^3 \preceq F(x_0^3, \ldots\ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ F(x_0^n, x_0^1, x_0^2, \ldots\ldots\ldots, x_0^{n-1}) \preceq gx_0^n; \end{gathered} \tag{3.4.1}$$

(vi) there exists $\phi \in \Phi$ such that

$$d\big(F(U), F(V)\big) \leq \phi\big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gy^i)\big), \tag{3.4.2}$$

for all $U = (x^1, x^2, x^3, \ldots\ldots, x^n)$, $V = (y^1, y^2, y^3, \ldots\ldots, y^n) \in X^n$, with $gx^1 \preceq gy^1$, $gy^2 \preceq gx^2$, $gx^3 \preceq gy^3, \ldots\ldots\ldots, gy^n \preceq gx^n$. Then F and g have an n-tupled coincidence point.

Proof. In view of hypothesis $F(X^n) \subseteq g(X)$, we construct the sequences $\{x_m^1\}, \{x_m^2\}$, $\{x_m^3\}, \ldots, \{x_m^n\}$ in X as follows:

$$\begin{cases} gx_{m+1}^1 = F(x_m^1, x_m^2, x_m^3, \ldots, x_m^n) \\ gx_{m+1}^2 = F(x_m^2, x_m^3, \ldots, x_m^n, x_m^1) \\ gx_{m+1}^3 = F(x_m^3, \ldots, x_m^n, x_m^1, x_m^2) \\ \quad\vdots \\ gx_{m+1}^n = F(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}). \end{cases} \tag{3.4.3}$$

Now, we prove that for all $m \geq 0$,

$$gx_m^i \preceq gx_{m+1}^i \text{ if } i \text{ is odd and } gx_{m+1}^i \preceq gx_m^i \text{ if } i \text{ is even.} \tag{3.4.4}$$

We will prove it by the method of induction. Relations (3.4.1) and (3.4.3) implies that (3.4.4) holds for $m = 0$. Suppose that (3.4.4) holds for some $m > 0$. As F has the mixed g-monotone property, we have from (3.4.3) that

$$\begin{aligned}
gx^1_{m+1} = F(x^1_m, x^2_m, x^3_m, ..., x^n_m) &\preceq F(x^1_{m+1}, x^2_m, x^3_m, ..., x^n_m)\\
&\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_m, ..., x^n_m)\\
&\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_m)\\
&\quad\vdots\\
&\preceq F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1})\\
&= gx^1_{m+2},
\end{aligned}$$

$$\begin{aligned}
gx^2_{m+2} = F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}) &\preceq F(x^2_m, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1})\\
&\preceq F(x^2_m, x^3_m, ..., x^n_{m+1}, x^1_{m+1})\\
&\quad\vdots\\
&\preceq F(x^2_m, x^3_m, ..., x^n_m, x^1_{m+1})\\
&\preceq F(x^2_m, x^3_m, ..., x^n_m, x^1_m)\\
&= gx^2_{m+1}.
\end{aligned}$$

Also for the same reason,

$$gx^3_{m+1} = F(x^3_m, ..., x^n_m, x^1_m, x^2_m) \preceq F(x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1}, x^2_{m+1}) = gx^3_{m+2},$$

$$\vdots$$

$$gx^n_{m+2} = F(x^n_{m+1}, x^1_{m+1}, x^2_{m+1}, ..., x^{n-1}_{m+1}) \preceq F(x^n_m, x^1_m, x^2_m, ..., x^{n-1}_m) = gx^n_{m+1}.$$

Hence, by mathematical induction it follows that (3.4.4) holds for all m. We define a sequence $\delta_m \geq 0$ as follows:

$$\delta_m = d(gx^1_m, gx^1_{m+1}) + d(gx^2_m, gx^2_{m+1}) + ... + d(gx^n_m, gx^n_{m+1}).$$

In case, $\delta_{m_0} = 0$, for some $m_0 \in \mathbb{N} \cup \{0\}$, we have

$$d(gx^1_{m_0}, gx^1_{m_0+1}) = d(gx^2_{m_0}, gx^2_{m_0+1}) = ... = d(gx^n_{m_0}, gx^n_{m_0+1}) = 0.$$

Consequently on using (3.4.3), we get

$$\begin{cases}
gx^1_{m_0} = gx^1_{m_0+1} = F(x^1_{m_0}, x^2_{m_0}, x^3_{m_0}, ..., x^n_{m_0})\\
gx^2_{m_0} = gx^2_{m_0+1} = F(x^2_{m_0}, x^3_{m_0}, ..., x^n_{m_0}, x^1_{m_0})\\
gx^3_{m_0} = gx^3_{m_0+1} = F(x^3_{m_0}, ..., x^n_{m_0}, x^1_{m_0}, x^2_{m_0})\\
\quad\vdots\\
gx^n_{m_0} = gx^n_{m_0+1} = F(x^n_{m_0}, x^1_{m_0}, x^2_{m_0}, ..., x^{n-1}_{m_0}).
\end{cases}$$

So that $(x^1_{m_0}, x^2_{m_0}, x^3_{m_0}, ..., x^n_{m_0})$ is an n-tupled coincidence point of F and g and hence we are done. Otherwise, if $\delta_m > 0$, for all m then we have to show that for each n and all m

$$d(gx^n_{m+1}, gx^n_{m+2}) \leq \phi\big(\frac{\delta_m}{n}\big). \tag{3.4.5}$$

On using (3.4.2), (3.4.3) and (3.4.4), we obtain

$$\begin{aligned} d(gx^1_{m+1}, gx^1_{m+2}) &= d\big(F(x^1_m, x^2_m, x^3_m, ..., x^n_m), F(x^1_{m+1}, x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1})\big) \\ &\leq \phi\big(\tfrac{1}{n}\sum_{i=1}^{n} d(g(x^i_m), g(x^i_{m+1}))\big) \\ &= \phi\big(\tfrac{\delta_m}{n}\big). \end{aligned}$$

Similarly, we can inductively write

$$d(gx^2_{m+1}, gx^2_{m+2}) \leq \phi\big(\tfrac{\delta_m}{n}\big),$$

$$\vdots$$

$$d(gx^n_{m+1}, gx^n_{m+2}) \leq \phi\big(\tfrac{\delta_m}{n}\big).$$

Hence (3.4.5) holds for each n and for all m. On taking summation of (3.4.5) over i, we get

$$\sum_{i=1}^{n} d(gx^i_{m+1}, gx^i_{m+2}) \leq n\phi\big(\tfrac{\delta_m}{n}\big),$$

so that

$$\delta_{m+1} \leq n\phi\big(\frac{\delta_m}{n}\big). \tag{3.4.6}$$

Since $\phi(t) < t$ for all $t > 0$, therefore $\delta_{m+1} < \delta_m$ for all m so that $\{\delta_m\}$ is a decreasing sequence. Since it is bounded below (as $\delta_m > 0$), there is some $\delta \geq 0$ such that

$$\lim_{m\to\infty} \delta_m = \delta.$$

Now, we show that $\delta = 0$. Suppose on contrary that $\delta > 0$. Taking the limit of both sides of (3.4.6) as $m \to \infty$ and keeping in mind our supposition that $\lim\limits_{r\to t^+} \phi(r) < t$ for all $t > 0$, we have

$$\delta = \lim_{m\to\infty} \delta_{m+1} \leq \lim_{m\to\infty} n\,\phi\big(\frac{\delta_m}{n}\big) = n \lim_{\delta_m\to\delta+} \phi\big(\frac{\delta_m}{n}\big) < n\big(\frac{\delta}{n}\big) = \delta,$$

which is a contradiction so that $\delta = 0$ yielding thereby

$$\lim_{m\to\infty} \left(d(gx^1_m, gx^1_{m+1}) + d(gx^2_m, gx^2_{m+1}) + ... + d(gx^n_m, gx^n_{m+1})\right) = 0. \tag{3.4.7}$$

Next we show that $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ are Cauchy sequences. If possible suppose that at least one of $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{l(k)\}$ such that for all positive integers k with $m(k) > l(k) \geq k$

$$t_k = \sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{m(k)}) \geq \epsilon. \tag{3.4.8}$$

We may choose $m(k)$ corresponding to $l(k)$ such that it is the smallest integer satisfying (3.4.8) and $m(k) > l(k) \geq k$. Hence

$$\sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{m(k)-1}) < \epsilon. \tag{3.4.9}$$

On using (3.4.8), (3.4.9) and the triangle inequality, we have

$$\begin{aligned}\epsilon \leq t_k &\leq \sum_{i=1}^{n} \left(d(gx^i_{l(k)}, gx^i_{m(k)-1}) + d(gx^i_{m(k)-1}, gx^i_{m(k)})\right)\\ &= \sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{m(k)-1}) + \delta_{m(k)-1} < \epsilon + \delta_{m(k)-1}.\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.4.7), we have

$$\lim_{k\to\infty} t_k = \epsilon. \tag{3.4.10}$$

As earlier, on using the triangular inequality, we have

$$\begin{aligned}t_k &= \sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{m(k)})\\ &\leq \sum_{i=1}^{n} \left(d(gx^i_{l(k)}, gx^i_{l(k)+1}) + d(gx^i_{l(k)+1}, gx^i_{m(k)+1}) + d(gx^i_{m(k)+1}, gx^i_{m(k)})\right)\\ &= \sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{l(k)+1}) + \sum_{i=1}^{n} d(gx^i_{l(k)+1}, gx^i_{m(k)+1}) + \sum_{i=1}^{n} d(gx^i_{m(k)+1}, gx^i_{m(k)})\\ &\leq \delta_{l(k)} + \delta_{m(k)} + \sum_{i=1}^{n} d(gx^i_{l(k)+1}, gx^i_{m(k)+1}).\end{aligned} \tag{3.4.11}$$

Now, we have to show that for each n and for all m

$$d(gx^n_{l(k)+1}, gx^n_{m(k)+1}) \leq \phi(\frac{t_k}{n}). \tag{3.4.12}$$

As $l(k) < m(k)$, on using (3.4.4), we get

$$gx^i_{l(k)} \preceq gx^i_{m(k)} \text{ if } i \text{ is odd and } gx^i_{m(k)} \preceq gx^i_{l(k)} \text{ if } i \text{ is even.} \tag{3.4.13}$$

On using (3.4.2), (3.4.3) and (3.4.13), we have

$$\begin{aligned} d(gx^1_{l(k)+1}, gx^1_{m(k)+1}) &= d\big(F(x^1_{l(k)}, x^2_{l(k)}, x^3_{l(k)}, ..., x^n_{l(k)}), F(x^1_{m(k)}, x^2_{m(k)}, x^3_{m(k)}, ..., x^n_{m(k)})\big) \\ &\leq \phi\big(\tfrac{1}{n}\sum_{i=1}^{n} d(gx^i_{l(k)}, gx^i_{m(k)})\big) \\ &= \phi\big(\tfrac{t_k}{n}\big). \end{aligned}$$

Similarly, we can show that

$$d(gx^2_{l(k)+1}, gx^2_{m(k)+1}) \leq \phi\big(\tfrac{t_k}{n}\big),$$

$$\vdots$$

$$d(gx^n_{l(k)+1}, gx^n_{m(k)+1}) \leq \phi\big(\tfrac{t_k}{n}\big).$$

Hence, (3.4.12) holds for each n and for all m. Making use of (3.4.12) in (3.4.11), we get

$$t_k \leq \delta_{l(k)} + \delta_{m(k)} + n\phi\big(\frac{t_k}{n}\big).$$

Letting $k \to \infty$ in the above inequality and using (3.4.7) and (3.4.10), we have

$$\epsilon \leq n \lim_{k\to\infty} \phi\big(\frac{t_k}{n}\big) = n \lim_{t_k\to\epsilon^+} \phi\big(\frac{t_k}{n}\big) < n\big(\frac{\epsilon}{n}\big) = \epsilon,$$

which is a contradiction. Therefore $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ are Cauchy sequences in $g(X)$. In view of assumption (i) firstly we assume that $g(X)$ is complete, then there exist $x^1, x^2, ..., x^n \in X$ such that

$$\lim_{m\to\infty} gx^1_m = gx^1, \lim_{m\to\infty} gx^2_m = gx^2, ..., \lim_{m\to\infty} gx^n_m = gx^n.$$

On the other hand if $F(X^n)$ is complete then in view of (3.4.3), $\{F(x^1_m, x^2_m, ..., x^n_m)\}, \{F(x^2_m, ..., x^n_m, x^1_m)\}, ..., \{F(x^n_m, x^1_m, ..., x^{n-1}_m)\}$ are Cauchy sequences in $F(X^n)$. Hence by using completeness of $F(X^n)$ and assumption (ii), there exist $x^1, x^2, ..., x^n \in X$ such that

$$\lim_{m\to\infty} gx^1_{m+1} = \lim_{m\to\infty} F(x^1_m, x^2_m, ..., x^n_m) = gx^1$$

$$\lim_{m\to\infty} gx_{m+1}^2 = \lim_{m\to\infty} F(x_m^2, ..., x_m^n, x_m^1) = gx^2$$

$$\vdots$$

$$\lim_{m\to\infty} gx_{m+1}^n = \lim_{m\to\infty} F(x_m^n, x_m^1, ..., x_m^{n-1}) = gx^n.$$

Hence in both the cases, we have

$$\lim_{m\to\infty} gx_m^1 = gx^1, \lim_{m\to\infty} gx_m^2 = gx^2, ..., \lim_{m\to\infty} gx_m^n = gx^n. \tag{3.4.14}$$

Now, we show that F and g have an n-tupled coincidence point. To accomplish this we use assumption (iv), firstly we suppose that F and g are continuous. By Lemma 3.4.1, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. Without loss of generality, in view of $g(E) = g(X)$, we can consider $x_m^1, x_m^2, ..., x_m^n, x_1, x_2, ..., x_n \in E$.
Define $G : [g(E)]^n \to X$ by

$$G(ge^1, ge^2, ..., ge^n) = F(e^1, e^2, ..., e^n), \quad \text{for all } ge^1, ge^2, ..., ge^n \in g(E).$$

As $g : E \to X$ is one-to-one, G is well defined. Since F and g are continuous, it follows that G is continuous. Now, we have

$$\begin{aligned} F(x^1, x^2, ..., x^n) &= G(gx^1, gx^2, ..., gx^n) \\ &= \lim_{m\to\infty} G(gx_m^1, gx_m^2, ..., gx_m^n) \\ &= \lim_{m\to\infty} F(x_m^1, x_m^2, ..., x_m^n) \\ &= \lim_{m\to\infty} g(x_{m+1}^1) = gx_1. \end{aligned}$$

Similarly, we have
$gx_2 = F(x^2, ..., x^n, x^1), ..., gx_n = F(x^n, x^1, ..., x^{n-1})$.
Thus, $(x^1, x^2, ..., x^n)$ is an n-tupled coincidence point of F and g.
On the other hand, suppose that $(gX, d, \preceq)$ has MCB property. Since $\{gx_m^i\}$ is increasing and decreasing according as i is odd or even, respectively and for each i, $gx_m^i \to gx^i$ as $m \to \infty$, we have

$$gx_m^i \preceq gx^i \text{ if } i \text{ is odd and } gx^i \preceq gx_m^i \text{ if } i \text{ is even.} \tag{3.4.15}$$

On using (3.4.3), (3.4.15) and assumption (vi), we have

$$d\big(gx^1_{m+1}, F(x^1, x^2, ..., x^n)\big) = d\big(F(x^1_m, x^2_m, ..., x^n_m), F(x^1, x^2, ..., x^n)\big)$$
$$\leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i_m, gx^i)\Big). \tag{3.4.16}$$

Then we claim that

$$d\big(gx^1_{m+1}, F(x^1, x^2, ..., x^n)\big) \leq \frac{1}{n}\sum_{i=1}^{n} d(gx^i_m, gx^i), \qquad \forall m \in \mathbb{N}. \tag{3.4.17}$$

Now, there arises two different possibilities so that we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$ such that
(*i*) $d(gx^i_m, gx^i) = 0, \qquad \forall m \in \mathbb{N}^0$
(*ii*) $d(gx^i_m, gx^i) > 0, \qquad \forall m \in \mathbb{N}^+$.

In case (*i*), on using Lemma 3.4.2, we get

$$d\big(gx^1_{m+1}, F(x^1, x^2, ..., x^n)\big) = d\big(F(x^1_m, x^2_m, ..., x^n_m), F(x^1, x^2, ..., x^n)\big) = 0,$$

this implies that (3.4.17) holds for all $m \in \mathbb{N}^0$.
Now, in case (*ii*), by using the property of ϕ, we get

$$d\big(gx^1_{m+1}, F(x^1, x^2, ..., x^n)\big) \leq \phi\big(\tfrac{1}{n}\sum_{i=1}^{n} d(gx^i_m, gx^i)\big)$$
$$< \tfrac{1}{n}\sum_{i=1}^{n} d(gx^i_m, gx^i), \qquad \forall m \in \mathbb{N}^+,$$

this implies that (3.4.17) holds for all $m \in \mathbb{N}^+$. Thus, (3.4.17) holds for all $m \in \mathbb{N}$. Taking $m \to \infty$ in (3.4.17) and on using (3.4.3) and (3.4.14), we get

$$d\big(gx^1, F(x^1, x^2, ..., x^n)\big) = 0,$$

this implies that $gx^1 = F(x^1, x^2, ..., x^n)$.

Analogously, we can get

$$gx^2 = F(x^2, ..., x^n, x^1), ..., gx^n = F(x^n, x^1, ..., x^{n-1}).$$

Therefore, $(x^1, x^2, ..., x^n)$ is an n-tupled coincidence point of F and g.
This completes the proof.

Now, we present a result regarding the uniqueness of n-tupled point of coincidence.

Theorem 3.4.2. *In addition to the hypotheses of Theorem 3.4.1, suppose that for all* $(x^1, x^2, ..., x^n), (y^1, y^2, ..., y^n) \in X^n$, *there exists* $(z^1, z^2, ..., z^n) \in X^n$ *such that* $(gz^1, gz^2, ...,$

gz^n) is comparable to $(gx^1, gx^2, ..., gx^n)$ and $(gy^1, gy^2, ..., gy^n)$. Then F and g have a unique n-tupled point of coincidence.

Proof. In view of Theorem 3.4.1, the set of n-tupled coincidence points of F and g is non-empty. Assume that, $(x^1, x^2, ..., x^n)$ and $(y^1, y^2, ..., y^n)$ are two n-tupled coincidence points of F and g, *i.e.*,

$$\begin{cases} gx^1 = F(x^1, x^2, ..., x^n), \ gy^1 = F(y^1, y^2, ..., y^n), \\ gx^2 = F(x^2, ..., x^n, x^1), \ gy^2 = F(y^2, ..., y^n, y^1), \\ \vdots \\ gx^n = F(x^n, x^1, ..., x^{n-1}), \ gy^n = F(y^n, y^1, ..., y^{n-1}). \end{cases} \tag{3.4.18}$$

We have to show that

$$(gx^1, gx^2, ..., gx^n) = (gy^1, gy^2, ..., gy^n). \tag{3.4.19}$$

By one of the assumptions, there exists $(z^1, z^2, ..., z^n) \in X^n$ such that $(gz^1, gz^2, ..., gz^n)$ is comparable to $(gx^1, gx^2, ..., gx^n)$ and $(gy^1, gy^2, ..., gy^n)$, i.e., either $(gz^1, gz^2, ..., gz^n) \lesssim (gx^1, gx^2, ..., gx^n)$ or $(gx^1, gx^2, ..., gx^n) \lesssim (gz^1, gz^2, ..., gz^n)$, and same argument will be true for $(gz^1, gz^2, ..., gz^n)$ and $(gy^1, gy^2, ..., gy^n)$. We suppose $(gx^1, gx^2, ..., gx^n) \lesssim (gz^1, gz^2, ..., gz^n)$ (the other case is similar), which implies that

$$gx^1 \preceq gz^1, gz^2 \preceq gx^2, ..., gz^n \preceq gx^n.$$

Put $z_0^1 = z^1, z_0^2 = z^2, ..., z_0^n = z^n$. Since $F(X^n) \subseteq g(X)$, on the lines similar to the proof of Theorem 3.4.1, we can inductively define sequences $\{z_m^1\}, \{z_m^2\}, ..., \{z_m^n\}$ such that

$$\begin{cases} gz_{m+1}^1 = F(z_m^1, z_m^2, z_m^3, ..., z_m^n) \\ gz_{m+1}^2 = F(z_m^2, z_m^3, ..., z_m^n, z_m^1) \\ \vdots \\ gz_{m+1}^n = F(z_m^n, z_m^1, z_m^2, ..., z_m^{n-1}), \text{ for all } m. \end{cases} \tag{3.4.20}$$

Now, for all $m \geq 0$, it can be easily seen that

$$\begin{cases} gx^1 \preceq gz_m^1 \preceq gz_{m+1}^1 \\ gz_{m+1}^2 \preceq gz_m^2 \preceq gx^2 \\ \vdots \\ gz_{m+1}^n \preceq gz_m^n \preceq gx^n. \end{cases} \tag{3.4.21}$$

Denote

$$\gamma_m = d(gx^1, gz_m^1) + d(gx^2, gz_m^2) + ... + d(gx^n, gz_m^n), \tag{3.4.22}$$

then we have to show that

$$\lim_{m\to\infty} \gamma_m = 0. \tag{3.4.23}$$

Now, two cases arise. Firstly, suppose that $\gamma_{m_0} = 0$, for some $m_0 \in \mathbb{N}$ then

$$d(gx^1, gz^1_{m_0}) = d(gx^2, gz^2_{m_0}) = ... = d(gx^n, gz^n_{m_0}) = 0.$$

On using Lemma 3.4.2, we obtain

$$\begin{cases} F(x^1, x^2, ..., x^n) = F(z^1_{m_0}, z^2_{m_0}, ..., z^n_{m_0}), \\ F(x^2, ..., x^n, x^1) = F(z^2_{m_0}, ..., z^n_{m_0}, z^1_{m_0}), \\ \vdots \\ F(x^n, x^1, ..., x^{n-1}) = F(z^n_{m_0}, z^1_{m_0}, ..., z^{n-1}_{m_0}). \end{cases}$$

Consequently, on using (3.4.18) and (3.4.20), we get

$$\begin{cases} d(gx^1, gz^1_{m_0+1}) = d(F(x^1, x^2, ..., x^n), F(z^1_{m_0}, z^2_{m_0}, ..., z^n_{m_0})) = 0, \\ d(gx^2, gz^2_{m_0+1}) = d(F(x^2, ..., x^n, x^1), F(z^2_{m_0}, ..., z^n_{m_0}, z^1_{m_0})) = 0, \\ \vdots \\ d(gx^n, gz^n_{m_0+1}) = d(F(x^n, x^1, ..., x^{n-1}), F(z^n_{m_0}, z^1_{m_0}, ..., z^{n-1}_{m_0})) = 0, \end{cases}$$

which on using (3.4.22) implies that $\gamma_{m_0+1} = 0$. Thus by induction, we get $\gamma_m = 0$, $\forall m \geq m_0$, yielding thereby $\lim\limits_{m\to\infty} \gamma_m = 0$. Hence in this case (3.4.23) is proved.

On the other hand suppose that $\gamma_m > 0$, for all $m \geq 1$. Then on using (3.4.18), (3.4.20), (3.4.21) and assumption (vi), we get

$$\begin{aligned} d(gx^1, gz^1_{m+1}) &= d(F(x^1, x^2, ..., x^n), F(z^1_m, z^2_m, ..., z^n_m)) \\ &\leq \phi(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gz^i_m)) = \phi(\frac{\gamma_m}{n}). \end{aligned}$$

Similarly, we have

$$d(gx^2, gz^2_{m+1}) \leq \phi(\frac{\gamma_m}{n}), ..., d(gx^n, gz^n_{m+1}) \leq \phi(\frac{\gamma_m}{n}).$$

On adding the above inequalities, we get

$$d(gx^1, gz^1_{m+1}) + d(gx^2, gz^2_{m+1}) + ... + d(gx^n, gz^n_{m+1}) \leq n\phi(\frac{\gamma_m}{n}),$$

so that

$$\gamma_{m+1} \leq n\phi(\frac{\gamma_m}{n}), \qquad \forall m \geq 1. \tag{3.4.24}$$

Since $\phi(t) < t$ for all $t > 0$, therefore $\gamma_{m+1} < n(\frac{\gamma_m}{n}) = \gamma_m$ for all $m \geq 1$, so that $\{\gamma_m\}$ is a decreasing sequence. Since it is bounded below (as $\gamma_m > 0$), there is some $\gamma \geq 0$ such that

$$\lim_{m\to\infty} \gamma_m = \gamma.$$

Now, we show that $\gamma = 0$. Suppose, on contrary that $\gamma > 0$. Taking the limit of both the sides of (3.4.24) as $m \to \infty$ and keeping in mind our supposition that $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0$, we have

$$\gamma = \lim_{m\to\infty} \gamma_{m+1} \leq n \lim_{m\to\infty} \phi\Big(\frac{\gamma_m}{n}\Big) = n \lim_{\gamma_m\to\gamma^+} \phi\Big(\frac{\gamma_m}{n}\Big) < n\Big(\frac{\gamma}{n}\Big) = \gamma,$$

which is a contradiction, so that $\gamma = 0$, yielding thereby $\lim_{m\to\infty} \gamma_m = 0$. Hence, in both the cases, (3.4.23) hold. Consequently from (3.4.23), we have

$$\lim_{m\to\infty} d(gx^1, gz^1_m) = 0, \lim_{m\to\infty} d(gx^2, gz^2_m) = 0, ..., \lim_{m\to\infty} d(gx^n, gz^n_m) = 0. \qquad (3.4.25)$$

Similarly, one can prove that

$$\lim_{m\to\infty} d(gy^1, gz^1_m) = 0, \lim_{m\to\infty} d(gy^2, gz^2_m) = 0, ..., \lim_{m\to\infty} d(gy^n, gz^n_m) = 0. \qquad (3.4.26)$$

By the triangular inequality, (3.4.25) and (3.4.26), we have

$$d(gx^1, gy^1) \leq d(gx^1, gz^1_m) + d(gz^1_m, gy^1) \to 0 \text{ as } m \to \infty$$

$$d(gx^2, gy^2) \leq d(gx^2, gz^2_m) + d(gz^2_m, gy^2) \to 0 \text{ as } m \to \infty$$

$$\vdots$$

$$d(gx^n, gy^n) \leq d(gx^n, gz^n_m) + d(gz^n_m, gy^n) \to 0 \text{ as } m \to \infty.$$

This implies that

$$gx^1 = gy^1, gx^2 = gy^2, ..., gx^n = gy^n.$$

Hence, (3.4.19) is proved.

Theorem 3.4.3. *In addition to the hypotheses of Theorem 3.4.2, suppose that the mappings F and g are w-compatible. Then F and g have a unique common n-tupled fixed point.*

Proof. Let $(x^1, x^2, ..., x^n) \in X^n$ be an n-tupled coincidence point of F and g, i.e.,

$$\begin{cases} gx^1 = F(x^1, x^2, ..., x^n), \\ gx^2 = F(x^2, ..., x^n, x^1), \\ \qquad \vdots \\ gx^n = F(x^n, x^1, ..., x^{n-1}). \end{cases}$$

Owing to w-compatibility of F and g, we have

$$\begin{cases} g(gx^1) = g(F(x^1, x^2, ..., x^n)) = F(gx^1, gx^2, ..., gx^n), \\ g(gx^2) = g(F(x^2, ..., x^n, x^1)) = F(gx^2, ..., gx^n, gx^1), \\ \vdots \\ g(gx^n) = g(F(x^n, x^1, ..., x^{n-1})) = F(gx^n, gx^1, ..., gx^{n-1}). \end{cases} \tag{3.4.27}$$

Write $gx^1 = z^1, gx^2 = z^2, ..., gx^n = z^n$. Then from (3.4.27), we have

$$\begin{cases} gz^1 = F(z^1, z^2, ..., z^n), \\ gz^2 = F(z^2, ..., z^n, z^1), \\ \vdots \\ gz^n = F(z^n, z^1, ..., z^{n-1}). \end{cases} \tag{3.4.28}$$

Thus, $(z^1, z^2, ..., z^n)$ is an n-tupled coincidence point. Then owing to (3.4.19) with $y^1 = z^1, y^2 = z^2, ..., y^n = z^n$, it follows that $gz^1 = gx^1, gz^2 = gx^2, ..., gz^n = gx^n$, *i.e.*,

$$gz^1 = z^1, gz^2 = z^2, ..., gz^n = z^n. \tag{3.4.29}$$

Using (3.4.28) and (3.4.29), we have

$$\begin{cases} z^1 = gz^1 = F(z^1, z^2, ..., z^n), \\ z^2 = gz^2 = F(z^2, ..., z^n, z^1), \\ \vdots \\ z^n = gz^n = F(z^n, z^1, ..., z^{n-1}). \end{cases}$$

Hence, $(z^1, z^2, ..., z^n)$ is a common n-tupled fixed point of F and g. To prove the uniqueness, assume that $(w^1, w^2, ..., w^n)$ is another common n-tupled fixed point of F and g. Then by (3.4.19), we have $w^1 = gw^1 = gz^1 = z^1, w^2 = gw^2 = gz^2 = z^2, ..., w^n = gw^n = gz^n = z^n$. This completes the proof.

3.5 Illustrative Example

Now, we furnish the following example to demonstrate the validity of Theorem 3.4.1.

Example 3.5.1. Let $X = \mathbb{R}$, endowed with the usual metric and usual order. Then $(X, d, \preceq)$ is an ordered metric space. Define mappings $F : X^n \to X$ and $g : X \to X$ by $F(x^1, x^2, ..., x^n) = 2$, for all $(x^1, x^2, ..., x^n) \in X^n$ and $g(x) = x - 3$, for all $x \in X$. Since

$$g(F(x^1, x^2, ..., x^n)) = g(2) = -1 \neq 2 = F(gx^1, gx^2, ..., gx^n),$$

for all $x^1, x^2, ..., x^n \in X$. Therefore, the mappings F and g do not satisfy the commutativity condition. We show that F and g are not compatible. Let $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ are the sequences in X such that

$$\begin{cases} \lim\limits_{m\to\infty} F(x_m^1, x_m^2, ..., x_m^n) = \lim\limits_{m\to\infty} gx_m^1 = a_1, \\ \lim\limits_{m\to\infty} F(x_m^2, ..., x_m^n, x_m^1) = \lim\limits_{m\to\infty} gx_m^2 = a_2, \\ \qquad\qquad \vdots \\ \lim\limits_{m\to\infty} F(x_m^n, x_m^1, ..., x_m^{n-1}) = \lim\limits_{m\to\infty} gx_m^n = a_n. \end{cases}$$

Then obviously, $a_1 = a_2 = ... = a_n = 2$. Further, it follows that

$$\begin{cases} \lim\limits_{m\to\infty} d\big(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)\big) = 2 \neq 0, \\ \lim\limits_{m\to\infty} d\big(g(F(x_m^2, ..., x_m^n, x_m^1)), F(gx_m^2, ..., gx_m^n, gx_m^1)\big) = 2 \neq 0, \\ \qquad\qquad \vdots \\ \lim\limits_{m\to\infty} d\big(g(F(x_m^n, x_m^1, ..., x_m^{n-1})), F(x_m^n, x_m^1, ..., x_m^{n-1})\big) = 2 \neq 0. \end{cases}$$

Hence, the mappings F and g are not compatible. Thus the results in [91] and [53] cannot be applied to these functions. By simple calculations we can show that, all the conditions of Theorem 3.4.1 are satisfied and $(5, 5, ..., 5)$ is an n-tupled coincidence point of F and g.

Chapter 4

Multi-Tupled Fixed Point Theorems Without Mixed Monotone Property

4.1 Introduction

As mentioned in the preceding chapter the appearance of Ran and Reuring's theorem has inspired intense research activity on and around the theorem which continues to flourish. Consequently, coupled, tripled, quadrupled and n-tupled fixed point results has been presented in recent times. Here, it can be pointed out that the most important hypothesis of such theorems is mixed monotone property.

In 2012, Doric *et al.* [57] established coupled fixed point results without the mixed monotone property (abbreviated as, MMP). They showed that a mixed monotone property in coupled fixed point results for mappings in ordered metric spaces can be replaced by another property which is often easy to check. In particular, it is automatically satisfied in the case of a totally ordered space, the case which is important in applications. Hence, such results can be applied to a relatively wider class of problems. For similar results, one can be referred to [34, 35, 169, 168].

The purpose of this chapter is to prove some n-tupled coincidence point results for mappings without the mixed monotone property. Section 4.2 mainly contains relevant preliminary concepts and some known results. In Section 4.3, we rectify some discrepancies observed in recent results of Doric *et al.* [57], which also applies to corresponding results of Chandok *et al.* [34], Chandok and Tas [35] and some other results of this type.

The contents of this chapter are based on two research papers. One of them has been published in British Jour. Math. Comput. Sci., 4(5), 735-748, (2014) while the other one is submitted in Miskolc Math. Notes.

In Section 4.4, we establish some n-tupled coincidence point results in ordered metric spaces for a pair of mappings without mixed monotone property satisfying contractive condition of rational type. Also, we present a result on the existence and uniqueness of common n-tupled fixed point. In the last and final section, we furnish an illustrative example to demonstrate the main result proved in Section 4.4.

4.2 Preliminaries

Throughout this chapter, n stands for a fixed natural number greater than 1. We assume X to be a non-empty set and $X^n = X \times X \times .... \times X$ (n times). Recall that a triplet $(X, d, \preceq)$ is called an ordered metric space if X is a non-empty set, d is a metric on X and $\preceq$ is a partial order on X and in addition, if d is a complete metric on X, then we say that $(X, d, \preceq)$ is ordered complete metric space [154]. Two elements x and y are said to be comparable if either $x \preceq y$ or $y \preceq x$ and we denote it as $x \prec\succ y$.

Doric *et al.* [57] observed that the mixed monotone property (resp. mixed g-monotone property) in coupled fixed (resp. coincidence) point result for mappings in ordered metric space is not necessary and hence they replaced mixed g-monotone property, for a pair of mappings $F : X^2 \to X$ and $g : X \to X$, by the following alternating condition:

$$\text{If } x, y, u, v \in X \text{ are such that } gx \prec\succ F(x, y) = gu, \text{ then } F(x, y) \prec\succ F(u, v). \quad (4.2.1)$$

Using this property, Doric *et al.* [57] proved the following coincidence point theorem.

Theorem 4.2.1. [57] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^2 \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) g is continuous and $g(X)$ is closed;
(ii) $F(X^2) \subset g(X)$ and g and F are compatible;
(iii) g and F satisfy property (4.2.1);
(iv) there exist $x_0, y_0 \in X$ such that $gx_0 \prec\succ F(x_0, y_0)$ and $gy_0 \prec\succ F(y_0, x_0)$;
(v) there exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$, with $gx \prec\succ gu$ and $gy \prec\succ gv$, satisfies,

$$d\big(F(x, y), F(u, v)\big) \leq k \max \big\{d(gx, gu), d(gy, gv)\big\};$$

(vi) (a)F is continuous or (b) if $x_m \to x$ when $m \to \infty$ in X, then $x_m \prec\succ x$ for m sufficiently large.
Then there exist $x, y \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$, i.e., F and g have a coupled coincidence point $(x, y) \in X^2$.

T-9602 Acc. No. .... Muslim University

In 2013, Chandok *et al.* [34] proved the following result:

Theorem 4.2.2. [34] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^2 \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) g is continuous and $g(X)$ is closed;
(ii) $F(X^2) \subseteq g(X)$ and g and F are compatible;
(iii) g and F satisfy property (4.2.1);
(iv) there exist $x_0, y_0 \in X$ such that $gx_0 \prec\succ F(x_0, y_0)$ and $gy_0 \prec\succ F(y_0, x_0)$;
(v) there exists $\alpha \in [0, 1)$ such that for all $x, y, u, v \in X$, with $gx \prec\succ gu$ and $gy \prec\succ gv$, satisfies,

$$\begin{aligned} d\big(F(x,y), F(u,v)\big) \leq \alpha \max \Big\{ & d(gx, gu), d(gy, gv), \frac{d\big(gx, F(x,y)\big)d\big(gu, F(u,v)\big)}{d(gx, gu)}, \\ & \frac{d\big(gx, F(u,v)\big)d\big(gu, F(x,y)\big)}{d(gx, gu)}, \frac{d\big(gy, F(y,x)\big)d\big(gv, F(v,u)\big)}{d(gy, gv)}, \\ & \frac{d\big(gy, F(v,u)\big)d\big(gv, F(y,x)\big)}{d(gy, gv)} \Big\}; \end{aligned}$$

(vi) F is continuous.
Then there exist $x, y \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$, i.e., F and g have a coupled coincidence point $(x, y) \in X^2$.

In 2014, Chandok and Tas [35] proved the following result:

Theorem 4.2.3. [35] *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^2 \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) g is continuous and $g(X)$ is closed;
(ii) $F(X^2) \subseteq g(X)$ and g and F are compatible;
(iii) g and F satisfy property (4.2.1);
(iv) there exist $x_0, y_0 \in X$ such that $gx_0 \prec\succ F(x_0, y_0)$ and $gy_0 \prec\succ F(y_0, x_0)$;
(v) there exist a continuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ and $\phi(t) = t$ if and only if $t = 0$ and a non-negative real number L such that for all $x, y, u, v \in X$, with $gu \preceq gx$ and $gy \preceq gv$, satisfies,

$$\begin{aligned} d\big(F(x,y), F(u,v)\big) \leq \phi\big(\max\{d(gx, gu), d(gy, gv)\}\big) + L \min\big\{ & d\big(F(x,y), gu\big) \\ & d\big(F(u,v), gx\big), d\big(F(x,y), gx\big), d\big(F(u,v), gu\big)\big\}; \end{aligned}$$

(vi) (a)F is continuous or (b) if $x_m \to x$ when $m \to \infty$ in X, then $x_m \prec\succ x$ for m sufficiently large.

Then there exist $x, y \in X$ such that $F(x,y) = gx$ and $F(y,x) = gy$, i.e., F and g have a coupled coincidence point $(x,y) \in X^2$.

Doric *et al.* [57] also proved uniqueness theorem corresponding to Theorem 4.2.1 as follows:

Theorem 4.2.4. [57] *In addition to the hypotheses of Theorem 4.2.1 assume that (vii) for any two elements $(x,y),(u,v) \in X^2$ there exists $(w,z) \in X^2$ such that $\big(F(w,z), F(z,w)\big)$ is comparable to both $\big(F(x,y), F(y,x)\big)$ and $\big(F(u,v), F(v,u)\big)$.*
Then g and F have a unique common coupled fixed point, i.e., there exists a unique $(p,q) \in X^2$ such that $p = g(p) = F(p,q)$ and $q = g(q) = F(q,p)$.

In the similar way, uniqueness theorems are proved corresponding to Theorems 4.2.2 and 4.2.3 in the respective papers (see [34, 35]).

4.3 Some Observations

It is clear that the assumption of property (4.2.1) depends only on x, y and u and is independent of v. Due to this fact (4.2.1) must hold for all $v \in X$, unless we cannot apply this property in the proof of the main result of Doric *et al.* [see line 4 and 5 in the proof of Theorem 2.3, page 1804, in [57]]. But due to independent choice of v, the property (4.2.1) becomes very restrictive and very hard.

Hence, we modify property (4.2.1) by the following property (4.3.1), so that the proof does work.

If $x, y, u, v \in X$ are such that $gx \prec\succ F(x,y) = gu$ and $gy \prec\succ F(y,x) = gv$, then

$$F(x,y) \prec\succ F(u,v). \tag{4.3.1}$$

Clearly, the assumption of property (4.3.1) depends on x, y, u and v, therefore property (4.3.1) is more convenient than property (4.2.1). Moreover, if property (4.2.1) holds then property (4.3.1) automatically holds so that property (4.3.1) is weaker than property (4.2.1).

We can define such properties in the form of a terminology as follows:

Definition 4.3.1. Let X be a non-empty set and $F : X^2 \to X$, $g : X \to X$ two mappings. We say that F enjoys $g - WC$ (weakly comparable) property if g and F satisfies (4.3.1).

Theorem 4.3.1. *Theorem 4.2.1 remains true if we replace condition (iii) by the following condition:*
(iii)′ F enjoys $g - WC$ property.

Proof. The proof of above result is same as Theorem 4.3.1 (i.e., Theorem 2.3 in [57]) with only a slight change (on the lines 4-5 of the proof of Theorem 2.3 at page 1804 of [57]), which is given below:

$$gx_m = F(x_{m-1}, y_{m-1}) \text{ and } gy_m = F(y_{m-1}, x_{m-1}), \text{ for } m = 1, 2, ... \tag{4.3.2}$$

(See (2.4) in [57]).
By assumption (iv) and (4.3.2) (with $m = 1$), we get

$$gx_0 \prec\succ F(x_0, y_0) = gx_1 \text{ and } gy_0 \prec\succ F(y_0, x_0) = gy_1. \tag{4.3.3}$$

Hence, on using assumption (iii)′ and (4.3.2), we get

$$gx_1 = F(x_0, y_0) \prec\succ F(x_1, y_1) = gx_2.$$

Rewrite (4.3.3) as

$$gy_0 \prec\succ F(y_0, x_0) = gy_1 \text{ and } gx_0 \prec\succ F(x_0, y_0) = gx_1. \tag{4.3.4}$$

Again, using assumption (iii)′ and (4.3.2), we get

$$gy_1 = F(y_0, x_0) \prec\succ F(y_1, x_1) = gy_2.$$

Proceeding by induction, we get that

$$gx_{m-1} \prec\succ gx_m \text{ and } gy_{m-1} \prec\succ gy_m, \text{ for each } m \in \mathbb{N}.$$

After this (see line 6 of the proof of Theorem 2.3 at page 1804 of [57]), the proof is similar.

Theorem 4.3.2. *Theorem 4.2.2 remains true if we replace condition (iii) by the following condition:*
(iii)′ F enjoys $g - WC$ property.

Theorem 4.3.3. *Theorem 4.2.3 remains true if we replace condition (iii) by the following condition:*
(iii)′ F enjoys $g - WC$ property.

The proofs of Theorem 4.3.2 and Theorem 4.3.3 are same as Theorem 4.3.1.

Finally, we make some observations on Theorem 4.2.4 (i.e., Theorem 2.7 in [57]). In view of lines 11-12 of the proof of Theorem 2.7 (see [57]), authors conclude $gx \prec\succ gw_1$, $gy \prec\succ gz_1$ and thereafter, further claim that "and in a similar way, $gx \prec\succ gw_m$, $gy \prec\succ gz_m$." But it is not possible as relation $\prec\succ$ is not transitive in general. To avoid this error, we must add a new assumption in the hypotheses.

Definition 4.3.2. [9] A partially ordered set $(X, \preceq)$ is called sequentially chainable if for every sequence $\{x_m\} \subset X$ with $x_m \prec\succ x_{m+1}\ \forall\ m$, we have

$$x_m \prec\succ x_l\ \forall\ m, l.$$

Theorem 4.3.4. *Theorem 4.2.4 remains true if we add the following assumption:*
(viii) $(X, \preceq)$ *is sequentially chainable.*

In view of foregoing observations, we must add the above property in uniqueness theorems contained in [34, 35].

4.4 n-Tupled Fixed Point Theorems for Contractive Rational Type Condition

In this section, we extend property (4.3.1) for a pair of mappings $F : X^n \to X$ and $g : X \to X$ in the following ways:

If $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$, are such that

$$\begin{cases} gx^1 \prec\succ F(x^1, x^2, ..., x^n) = gy^1 \\ gx^2 \prec\succ F(x^2, ..., x^n, x^1) = gy^2 \\ \qquad\qquad \vdots \\ gx^n \prec\succ F(x^n, x^1, ..., x^{n-1}) = gy^n, \end{cases}$$

then

$$F(x^1, x^2, ..., x^n) \prec\succ F(y^1, y^2, ..., y^n). \tag{4.4.1}$$

We can define such properties in the form of a terminology as follows:

Definition 4.4.1. Let X be a non-empty set and $F : X^n \to X$, $g : X \to X$ two mappings. We say that F enjoys $g - WC$ (weakly comparable) property if g and F satisfies (4.4.1).

Notice that, by setting $n = 2$ in Definition 4.4.1, we deduce a coupled formulation of $g - WC$ property.

Using this property, we prove the following result:

Theorem 4.4.1. *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^n \to X$, $g : X \to X$ two mappings. Suppose that the following conditions hold:*
(i) g is continuous and $g(X)$ is closed;
(ii) $F(X^n) \subseteq g(X)$ and g and F are compatible;
(iii) F enjoys $g - WC$ property;
(iv) there exist $x_0^1, x_0^2, ..., x_0^n \in X$ such that

$$gx_0^1 \prec\succ F(x_0^1, x_0^2, ..., x_0^n),\ gx_0^2 \prec\succ F(x_0^2, ..., x_0^n, x_0^1), ...,\ gx_0^n \prec\succ F(x_0^n, x_0^1, ..., x_0^{n-1});$$

(v) there exists $\alpha \in [0, 1)$ such that for all $U = (x^1, x^2, ..., x^n)$, $V = (y^1, y^2, ..., y^n) \in X^n$, for which $gx^1 \prec\succ gy^1$, $gx^2 \prec\succ gy^2, ..., gx^n \prec\succ gy^n$, with $gx^1 \neq gy^1, gx^2 \neq gy^2, ..., gx^n \neq gy^n$, satisfies,

$$\begin{aligned} d(F(U), F(V)) \leq \alpha \max \Big\{ & d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n), \\ & \frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy^1, F(y^1, y^2, ..., y^n))}{d(gx^1, gy^1)}, \\ & \frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy^2, F(y^2, ..., y^n, y^1))}{d(gx^2, gy^2)}, ..., \\ & \frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy^n, F(y^n, y^1, ..., y^{n-1}))}{d(gx^n, gy^n)}, \\ & \frac{d(gx^1, F(y^1, y^2, ..., y^n))d(gy^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy^1)}, \\ & \frac{d(gx^2, F(y^2, ..., y^n, y^1))d(gy^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy^2)}, ..., \\ & \frac{d(gx^n, F(y^n, y^1, ..., y^{n-1}))d(gy^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy^n)} \Big\}; \end{aligned} \tag{4.4.2}$$

(vi) F is continuous.
Then there exist $x^1, x^2, ..., x^n \in X$ such that $F(x^1, x^2, ..., x^n) = gx^1$, $F(x^2, ..., x^n, x^1) = gx^2, ..., F(x^n, x^1, ..., x^{n-1}) = gx^n$, i.e., F and g have an n-tupled coincidence point $(x^1, x^2, ..., x^n) \in X^n$.

Proof. Using condition (ii), we construct sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ in X as follows:

$$\begin{cases} gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n) \\ gx_m^2 = F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1) \\ \quad\vdots \\ gx_m^n = F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}), \quad \text{for} \quad m \geq 1. \end{cases} \tag{4.4.3}$$

On using condition (iv) and (4.4.3) (with $m = 1$), we have

$$\begin{cases} gx_0^1 \prec\succ F(x_0^1, x_0^2, ..., x_0^n) = gx_1^1 \\ gx_0^2 \prec\succ F(x_0^2, ..., x_0^n, x_0^1) = gx_1^2 \\ \vdots \\ gx_0^n \prec\succ F(x_0^n, x_0^1, ..., x_0^{n-1}) = gx_1^n. \end{cases}$$

Hence, on applying (iii) and (4.4.3), we get

$$gx_1^1 = F(x_0^1, x_0^2, ..., x_0^n) \prec\succ F(x_1^1, x_1^2, ..., x_1^n) = gx_2^1.$$

Similarly, one can get

$$gx_1^2 = F(x_0^2, ..., x_0^n, x_0^1) \prec\succ F(x_1^2, ..., x_1^n, x_1^1) = gx_2^2$$

$$\vdots$$

$$gx_1^n = F(x_0^n, x_0^1, ..., x_0^{n-1}) \prec\succ F(x_1^n, x_1^1, ..., x_1^{n-1}) = gx_2^n.$$

Proceeding by induction, we get

$$gx_{m-1}^1 \prec\succ gx_m^1,\ gx_{m-1}^2 \prec\succ gx_m^2, ...,\ gx_{m-1}^n \prec\succ gx_m^n,$$

for each $m \geq 1$.

Now, from contractive condition (4.4.2), we have

$$\begin{aligned} d(gx_{m+1}^1, gx_m^1) &= d\big(F(x_m^1, x_m^2, ..., x_m^n), F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n)\big) \\ &\leq \alpha \max \Big\{ d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n), \\ &\qquad \frac{d(gx_m^1, F(x_m^1, x_m^2, ..., x_m^n))d(gx_{m-1}^1, F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n))}{d(gx_m^1, gx_{m-1}^1)}, \\ &\qquad \frac{d(gx_m^2, F(x_m^2, ..., x_m^n, x_m^1))d(gx_{m-1}^2, F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1))}{d(gx_m^2, gx_{m-1}^2)}, ..., \\ &\qquad \frac{d(gx_m^n, F(x_m^n, x_m^1, ..., x_m^{n-1}))d(gx_{m-1}^n, F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}))}{d(gx_m^n, gx_{m-1}^n)}, \\ &\qquad \frac{d(gx_m^1, F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n))d(gx_{m-1}^1, F(x_m^1, x_m^2, ..., x_m^n))}{d(gx_m^1, gx_{m-1}^1)}, \\ &\qquad \frac{d(gx_m^2, F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1))d(gx_{m-1}^2, F(x_m^2, ..., x_m^n, x_m^1))}{d(gx_m^2, gx_{m-1}^2)}, ..., \end{aligned}$$

$$\left.\frac{d(gx_m^n, F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}))d(gx_{m-1}^n, F(x_m^n, x_m^1, ..., x_m^{n-1}))}{d(gx_m^n, gx_{m-1}^n)}\right\}$$

$$= \alpha \max \left\{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),\right.$$

$$\frac{d(gx_m^1, gx_{m+1}^1)d(gx_{m-1}^1, gx_m^1)}{d(gx_m^1, gx_{m-1}^1)}, \frac{d(gx_m^2, gx_{m+1}^2)d(gx_{m-1}^2, gx_m^2)}{d(gx_m^2, gx_{m-1}^2)}, ...,$$

$$\frac{d(gx_m^n, gx_{m+1}^n)d(gx_{m-1}^n, gx_m^n)}{d(gx_m^n, gx_{m-1}^n)}, \frac{d(gx_m^1, gx_m^1)d(gx_{m-1}^1, gx_{m+1}^1)}{d(gx_m^1, gx_{m-1}^1)}$$

$$\left.\frac{d(gx_m^2, gx_m^2)d(gx_{m-1}^2, gx_{m+1}^2)}{d(gx_m^2, gx_{m-1}^2)}, ..., \frac{d(gx_m^n, gx_m^n)d(gx_{m-1}^n, gx_{m+1}^n)}{d(gx_m^n, gx_{m-1}^n)}\right\}$$

$$= \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$

$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}. \tag{4.4.4}$$

Similarly, we have

$$d(gx_{m+1}^2, gx_m^2) \leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$

$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}$$

$$\vdots$$

$$d(gx_{m+1}^n, gx_m^n) \leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$

$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}.$$

Let

$$\sigma_m = \max \{d(gx_{m+1}^1, gx_m^1), d(gx_{m+1}^2, gx_m^2), ..., d(gx_{m+1}^n, gx_m^n)\}.$$

Hence,

$$\max \{d(gx_{m+1}^1, gx_m^1), d(gx_{m+1}^2, gx_m^2), ..., d(gx_{m+1}^n, gx_m^n)\}$$

$$\leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n)\} = \alpha\sigma_{m-1}.$$

By induction, we get that

$$\max\{d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\} \leq \alpha^m \sigma_0.$$

It easily follows that for each $m, l \in \mathbb{N}$ with $l < m$ we have

$$d(gx_l^1, gx_m^1) \leq \frac{\alpha^l}{1-\alpha}\sigma_0,\ d(gx_l^2, gx_m^2) \leq \frac{\alpha^l}{1-\alpha}\sigma_0, ..., d(gx_l^n, gx_m^n) \leq \frac{\alpha^l}{1-\alpha}\sigma_0.$$

Therefore, $\{gx_m^1\}, \{gx_m^2\}, \ldots, \{gx_m^n\}$ are Cauchy sequences and since $g(X)$ is closed in a complete metric space, there exist $x^1, x^2, \ldots, x^n \in g(X)$ such that

$$\lim_{m\to\infty} gx_m^1 = \lim_{m\to\infty} F(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n) = x^1$$

$$\lim_{m\to\infty} gx_m^2 = \lim_{m\to\infty} F(x_{m-1}^2, \ldots, x_{m-1}^n, x_{m-1}^1) = x^2$$

$$\vdots$$

$$\lim_{m\to\infty} gx_m^n = \lim_{m\to\infty} F(x_{m-1}^n, x_{m-1}^1, \ldots, x_{m-1}^{n-1}) = x^n.$$

Compatibility of F and g implies that

$$\lim_{m\to\infty} d\big(g(F(x_m^1, x_m^2, \ldots, x_m^n)), F(gx_m^1, gx_m^2, \ldots, gx_m^n)\big) = 0$$

$$\lim_{m\to\infty} d\big(g(F(x_m^2, \ldots, x_m^n, x_m^1)), F(gx_m^2, \ldots, gx_m^n, gx_m^1)\big) = 0$$

$$\vdots$$

$$\lim_{m\to\infty} d\big(g(F(x_m^n, x_m^1, \ldots, x_m^{n-1})), F(gx_m^n, gx_m^1, \ldots, gx_m^{n-1})\big) = 0.$$

As F is continuous, therefore

$$F(gx_m^1, gx_m^2, \ldots, gx_m^n) \to F(x^1, x^2, \ldots, x^n)$$

$$F(gx_m^2, \ldots, gx_m^n, gx_m^1) \to F(x^2, \ldots, x^n, x^1)$$

$$\vdots$$

$$F(gx_m^n, gx_m^1, \ldots, gx_m^{n-1}) \to F(x^n, x^1, \ldots, x^{n-1}).$$

Using triangular inequality, we get

$$d\big(gx^1, F(gx_m^1, gx_m^2, \ldots, gx_m^n)\big)$$
$$\leq d\big(gx^1, g(F(x_m^1, x_m^2, \ldots, x_m^n))\big) + d\big(g(F(x_m^1, x_m^2, \ldots, x_m^n)), F(gx_m^1, gx_m^2, \ldots, gx_m^n)\big).$$

By taking $m \to \infty$ and using continuity of F and g, we have

$$d\big(gx^1, F(x^1, x^2, \ldots, x^n)\big) = 0, \text{ that is } gx^1 = F(x^1, x^2, \ldots, x^n),$$

and in a similar way, we have

$$gx^2 = F(x^2, \ldots, x^n, x^1), \ldots, \; gx^n = F(x^n, x^1, \ldots, x^{n-1}).$$

Thus, F and g have an n-tupled coincidence point.
This completes the proof.

Remark 4.4.1. On setting $n = 2$, Theorem 4.4.1 reduces to Theorem 4.3.2 with addition to assumption that $g(x) \neq g(u)$ and $g(y) \neq g(v)$. This is necessary, otherwise we cannot define the quotients '$\frac{d(g(x),F(x,y))d(gu,F(u,v))}{d(gx,gu)}$' etc.

Now, we shall prove the existence and uniqueness of a common n-tupled fixed point. Note that, if $(X, \preceq)$ is an ordered set, then we endow the product space X^n with the following partial order relation:

For $U = (x^1, x^2, ..., x^n),\ V = (y^1, y^2, ..., y^n) \in X^n$

$$U \lesssim V \Leftrightarrow x^1 \preceq y^1,\ y^2 \preceq x^2, ..., y^n \preceq x^n.$$

Theorem 4.4.2. *In addition to the hypotheses of Theorem 4.4.1, suppose that*
(vii) for every $(x^1, x^2, ..., x^n)$, $(z^1, z^2, ..., z^n) \in X^n$ *there exists,* $(y^1, y^2, ..., y^n) \in X^n$ *such that* $\big(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1})\big)$ *is comparable to both* $\big(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1})\big)$ *and* $\big(F(z^1, z^2, ..., z^n), F(z^2, ..., z^n, z^1), ..., F(z^n, z^1, ..., z^{n-1})\big)$;
(viii) $(X, \preceq)$ *is sequentially chainable;*
Then F and g have a unique common n-tupled fixed point.

Proof. From Theorem 4.4.1, the set of n-tupled coincidence points of F and g is non-empty. Suppose that, $(x^1, x^2, ..., x^n)$ and $(z^1, z^2, ..., z^n)$ are two n-tupled coincidence points of F and g, *i.e.*,

$$F(x^1, x^2, ..., x^n) = gx^1, \quad F(z^1, z^2, ..., z^n) = gz^1$$

$$F(x^2, ..., x^n, x^1) = gx^2, \quad F(z^2, ..., z^n, z^1) = gz^2$$

$$\vdots$$

$$F(x^n, x^1, ..., x^{n-1}) = gx^n, \quad F(z^n, z^1, ..., z^{n-1}) = gz^n.$$

We shall show that

$$gx^1 = gz^1, gx^2 = gz^2, ..., gx^n = gz^n.$$

Using condition (vii), there exists $(y^1, y^2, ..., y^n) \in X^n$ such that

$$\big(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1})\big),$$

is comparable to

$$\big(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1})\big),$$

and

$$\big(F(z^1,z^2,...,z^n),F(z^2,...,z^n,z^1),...,F(z^n,z^1,...,z^{n-1})\big).$$

Put $y_0^1=y^1,y_0^2=y^2,...,y_0^n=y^n$ and choose $y_1^1,y_1^2,...,y_1^n\in X$ such that

$$gy_1^1=F(y_0^1,y_0^2,...,y_0^n)$$

$$gy_1^2=F(y_0^2,...,y_0^n,y_0^1)$$

$$\vdots$$

$$gy_1^n=F(y_0^n,y_0^1,...,y_0^{n-1}).$$

Then similarly as in the proof of Theorem 4.4.1, we can inductively define sequences $\{gy_m^1\},\{gy_m^2\},...,\{gy_m^n\}$ such that

$$gy_m^1=F(y_{m-1}^1,y_{m-1}^2,...,y_{m-1}^n)$$

$$gy_m^2=F(y_{m-1}^2,...,y_{m-1}^n,y_{m-1}^1)$$

$$\vdots$$

$$gy_m^n=F(y_{m-1}^n,y_{m-1}^1,...,y_{m-1}^{n-1}),\ \forall\, m\in\mathbb{N}.$$

Further set $x_0^1=x^1,x_0^2=x^2,...,x_0^n=x^n$ and $z_0^1=z^1,z_0^2=z^2,...,z_0^n=z^n$ and in the same way, define the sequences $\{gx_m^1\},\{gx_m^2\},...,\{gx_m^n\}$ and $\{gz_m^1\},\{gz_m^2\},...,\{gz_m^n\}$. Then as in Theorem 4.4.1, we can show that

$$gx_m^1\to gx^1=F(x^1,x^2,...,x^n),\quad gz_m^1\to gz^1=F(z^1,z^2,...,z^n)$$

$$gx_m^2\to gx^2=F(x^2,...,x^n,x^1),\quad gz_m^2\to gz^2=F(z^2,...,z^n,z^1)$$

$$\vdots$$

$$gx_m^n\to gx^n=F(x^n,x^1,...,x^{n-1}),\quad gz_m^n\to gz^n=F(z^n,z^1,...,z^{n-1}),\ \forall\, m\in\mathbb{N}.$$

Since

$$\begin{aligned}&\big(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1})\big)\\&=(gx_1^1,gx_1^2,...,gx_1^n)=(gx^1,gx^2,...,gx^n),\end{aligned}$$

and

$$\begin{aligned}&\big(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1})\big)\\&\qquad= (gy_1^1, gy_1^2, ..., gy_1^n),\end{aligned}$$

are comparable. Then we have

$$gx^1 \prec\succ gy_1^1,\ gx^2 \prec\succ gy_1^2, ...,\ gx^n \prec\succ gy_1^n.$$

In view of condition (viii), we have

$$\begin{aligned}gy_m^1 &= F(y_{m-1}^1, y_{m-1}^2, ..., y_{m-1}^n) \prec\succ F(x^1, x^2, ..., x^n) = gx^1\\ gy_m^2 &= F(y_{m-1}^2, ..., y_{m-1}^n, y_{m-1}^1) \prec\succ F(x^2, ..., x^n, x^1) = gx^2\\ &\ \vdots\\ gy_m^n &= F(y_{m-1}^n, y_{m-1}^1, ..., y_{m-1}^{n-1}) \prec\succ F(x^n, x^1, ..., x^{n-1}) = gx^n.\end{aligned}$$

Thus, from (4.4.2) with $gx^1 \neq gy_m^1,\ gx^2 \neq gy_m^2, ..., gx^n \neq gy_m^n$, we have

$$\begin{aligned}d(gx^1, gy_{m+1}^1) &= d\big(F(x^1, x^2, ..., x^n), F(y_m^1, y_m^2, ..., y_m^n)\big)\\ &\leq \alpha \max\Bigg\{d(gx^1, gy_m^1), d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n),\\ &\qquad \frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy_m^1, F(y_m^1, y_m^2 .., ., y_m^n))}{d(gx^1, gy_m^1)},\\ &\qquad \frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy_m^2, F(y_m^2 .., ., y_m^n, y_m^1))}{d(gx^2, gy_m^2)}, ...,\\ &\qquad \frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy_m^n, F(y_m^n, y_m^1 .., ., y_m^{n-1}))}{d(gx^n, gy_m^n)},\\ &\qquad \frac{d(gx^1, F(y_m^1, y_m^2 .., ., y_m^n))d(gy_m^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy_m^1)},\\ &\qquad \frac{d(gx^2, F(y_m^2 .., ., y_m^n, y_m^1))d(gy_m^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy_m^2)}, ...,\\ &\qquad \frac{d(gx^n, F(y_m^n, y_m^1 .., ., y_m^{n-1}))d(gy_m^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy_m^n)}\Bigg\}\\ &= \alpha \max\{d(gx^1, gy_m^1), d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n),\\ &\qquad d(gx^1, gy_{m+1}^1), d(gx^2, gy_{m+1}^2), ..., d(gx^n, gy_{m+1}^n)\}.\end{aligned}$$

Similarly, we can prove that

$$d(gx^2, gy^2_{m+1}) \leq \alpha \max\{d(gx^2, gy^2_m), ..., d(gx^n, gy^n_m), d(gx^1, gy^1_m),$$
$$d(gx^2, gy^2_{m+1}), ..., d(gx^n, gy^n_{m+1}), d(gx^1, gy^1_{m+1})\}$$
$$\vdots$$
$$d(gx^n, gy^n_{m+1}) \leq \alpha \max\{d(gx^n, gy^n_m), d(gx^1, gy^1_m), ..., d(gx^{n-1}, gy^{n-1}_m),$$
$$d(gx^n, gy^n_{m+1}), d(gx^1, gy^1_{m+1}), ..., d(gx^{n-1}, gy^{n-1}_{m+1})\}.$$

Hence,

$$\max\{d(gx^1, gy^1_{m+1}), d(gx^2, gy^2_{m+1}), ..., d(gx^n, gy^n_{m+1})\}$$
$$\leq \alpha \max\{d(gx^1, gy^1_m), d(gx^2, gy^2_m), ..., d(gx^n, gy^n_m)\},$$

and by induction,

$$\max\{d(gx^1, gy^1_{m+1}), d(gx^2, gy^2_{m+1}), ..., d(gx^n, gy^n_{m+1})\}$$
$$\leq \alpha^m \max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}.$$

On taking limit $m \to \infty$, we get

$$\lim_{m\to\infty} d(gx^1, gy^1_{m+1}) = 0, \ \lim_{m\to\infty} d(gx^2, gy^2_{m+1}) = 0, ..., \lim_{m\to\infty} d(gx^n, gy^n_{m+1}) = 0.$$

Similarly, we can prove that

$$\lim_{m\to\infty} d(gz^1, gy^1_{m+1}) = 0, \ \lim_{m\to\infty} d(gz^2, gy^2_{m+1}) = 0, ..., \lim_{m\to\infty} d(gz^n, gy^n_{m+1}) = 0.$$

Finally, we have

$$d(gx^1, gz^1) \leq d(gx^1, gx^1_m) + d(gx^1_m, gz^1)$$
$$d(gx^2, gz^2) \leq d(gx^2, gx^2_m) + d(gx^2_m, gz^2)$$
$$\vdots$$
$$d(gx^n, gz^n) \leq d(gx^n, gx^n_m) + d(gx^n_m, gz^n).$$

Taking $m \to \infty$ in above inequalities, we get

$$d(gx^1, gz^1) = d(gx^2, gz^2) = ... = d(gx^n, gz^n) = 0,$$

i.e.,

$$gx^1 = gz^1, \ gx^2 = gz^2, ..., \ gx^n = gz^n.$$

Denote $gx^1 = p^1,\ gx^2 = p^2, ...,\ gx^n = p^n$, we have that

$$gp^1 = g(gx^1) = g(F(x^1, x^2, ..., x^n))$$
$$gp^2 = g(gx^2) = g(F(x^2, ..., x^n, x^1))$$
$$\vdots$$
$$gp^n = g(gx^n) = g(F(x^n, x^1, ..., x^{n-1})).$$

By the definition of sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$, we have

$$gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n)$$
$$gx_m^2 = F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1)$$
$$\vdots$$
$$gx_m^n = F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}),$$

and so

$$gx_m^1 \to F(x^1, x^2, ..., x^n),\ gx_m^2 \to F(x^2, ..., x^n, x^1), ...,\ gx_m^n \to F(x^n, x^1, ..., x^{n-1}).$$

Compatibility of F and g implies that

$$\lim_{m\to\infty} d\big(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)\big) \to 0,$$

i.e.,

$$g(F(x^1, x^2, ..., x^n)) = F(gx^1, gx^2, ..., gx^n).$$

Then $gp^1 = F(p^1, p^2, ..., p^n)$ and similarly, $gp^2 = F(p^2, ..., p^n, p^1), ...,\ gp^n = F(p^n, p^1, ..., p^{n-1}$ Thus, $(p^1, p^2, ..., p^n)$ is an n-tupled coincidence point of F and g. Hence, it follows that $gp^1 = gx^1,\ gp^2 = gx^2, ...,\ gp^n = gx^n$, *i.e.*, $gp^1 = p^1,\ gp^2 = p^2, ...,\ gp^n = p^n$. Hence,

$$p^1 = gp^1 = F(p^1, p^2, ..., p^n),$$
$$p^2 = gp^2 = F(p^2, ..., p^n, p^1),$$
$$\vdots$$
$$p^n = gp^n = F(p^n, p^1, ..., p^{n-1}).$$

Therefore, $(p^1, p^2, ..., p^n)$ is a common n-tupled fixed point of F and g. To prove the uniqueness, assume that $(q^1, q^2, ..., q^n)$ is another common n-tupled fixed point of F and g. Then as above we have

$$q^1 = gq^1 = gp^1 = p^1,$$

$$q^2 = gq^2 = gp^2 = p^2,$$
$$\vdots$$
$$q^n = gq^n = gp^n = p^n.$$

Hence, we get the result.

4.5 Illustrative Example

Now, we furnish the following example to demonstrate the validity of Theorem 4.4.1.

Example 4.5.1. Let $X = [0,1]$. Then $(X, d, \preceq)$ is a partially ordered set with the natural ordering $\preceq$ of real numbers and natural metric $d(x,y) = |x-y|$, for all $x, y \in X$. Define $g : X \to X$ by $g(x) = x^2$, for all $x \in X$ and $F : X^n \to X$ (wherein n is fixed and $n > 1$) by

$$F(x^1, x^2, ..., x^n) = \frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + ... + (x^{n-1})^2 + (x^n)^2}{2n},$$

for all $x^1, x^2, ..., x^n \in X$. All the conditions of Theorem 4.4.1 are satisfied, the contractive condition (for $\alpha = \frac{1}{2}$), follows from

$$\begin{aligned}
&d\big(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)\big) \\
&= d\left(\frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + ... + (x^n)^2}{2n}, \frac{(y^1)^2 + (y^2)^2 + (y^3)^2 + ... + (y^n)^2}{2n}\right) \\
&= \left|\frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + ... + (x^n)^2}{2n} - \frac{(y^1)^2 + (y^2)^2 + (y^3)^2 + ... + (y^n)^2}{2n}\right| \\
&= \left|\frac{((x^1)^2 - (y^1)^2) + ((x^2)^2 - (y^2)^2) + ((x^3)^2 - (y^3)^2) + ... + ((x^n)^2 - (y^n)^2)}{2n}\right| \\
&\leq \frac{|(x^1)^2 - (y^1)^2| + |(x^2)^2 - (y^2)^2| + |(x^3)^2 - (y^3)^2| + ... + |(x^n)^2 - (y^n)^2|}{2n} \\
&\leq \frac{1}{2n}\left[n \max\left\{|(x^1)^2 - (y^1)^2|, |(x^2)^2 - (y^2)^2|, |(x^3)^2 - (y^3)^2|, ..., |(x^n)^2 - (y^n)^2|\right\}\right] \\
&= \frac{1}{2}\left[\max\left\{|gx^1 - gy^1|, |gx^2 - gy^2|, |gx^3 - gy^3|, ..., |gx^n - gy^n|\right\}\right] \\
&= \frac{1}{2}\left[\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), d(gx^3, gy^3), ..., d(gx^n, gy^n)\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
\leq \alpha \max \Big\{ & d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n), \\
& \frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy^1, F(y^1, y^2, ..., y^n))}{d(gx^1, gy^1)}, \\
& \frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy^2, F(y^2, ..., y^n, y^1))}{d(gx^2, gy^2)}, ..., \\
& \frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy^n, F(y^n, y^1, ..., y^{n-1}))}{d(gx^n, gy^n)}, \\
& \frac{d(gx^1, F(y^1, y^2, ..., y^n))d(gy^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy^1)}, \\
& \frac{d(gx^2, F(y^2, ..., y^n, y^1))d(gy^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy^2)}, ..., \\
& \frac{d(gx^n, F(y^n, y^1, ..., y^{n-1}))d(gy^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy^n)} \Big\}.
\end{aligned}
$$

Hence, all the conditions of Theorem 4.4.1 are satisfied and $(0, 0, ..., 0)$ is an n-tupled coincidence point of F and g.

Chapter 5

System of Generalized Nonlinear Quasi-Variational-Like Inclusions

5.1 Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques. A useful and an important generalization of variational inequalities are variational inclusions. Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics, transportation equilibrium and engineering sciences (e.g. [3, 11, 16, 32, 64, 71]). Many efficient ways have been studied to find solutions for variational inclusions. Those methods include the projection method and its various forms, linear approximation, descent and Newtons method and the method based on auxiliary principle technique. The method based on the resolvent operator technique is a generalization of the projection method and has been widely used to solve variational inclusions. For further details, we refer to see [7, 108, 109, 110, 202, 209] and the references therein.

In 1997, Noor and Noor [149] introduced and studied resolvent equations by virtue of the resolvent operator technique and has established the equivalence between the mixed variational inequalities and the resolvent equations. The resolvent equations technique is being used to develop powerful and efficient numerical techniques for solving mixed (quasi) variational inequalities and related optimization problems.

The contents of this chapter are based on two research papers. One of them has been published in Chinese Jour. Math., 2014:957482 (2014) while the other one is accepted in International Mathematical Forum.

As generalizations of system of variational inequalities, Agarwal *et al.* [4] introduced a system of generalized mixed quasi-variational inclusions and investigated the sensitivity analysis of solutions for the system of generalized mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [105] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [61], Verma [198] and Fang *et al.* [63] introduced and studied new systems of variational inclusions involving H-monotone operators, A-monotone operators and (H, η)-monotone operators, respectively. Lan *et al.* [121] introduced and studied a new system of nonlinear A-monotone multivalued variational inclusions. For more details on the related research work of this field, we invoke the readers to consult [6, 31, 44, 56, 80, 108, 109, 110, 158, 199, 201, 202, 203, 206].

In this chapter, by using the resolvent operator associated with $H(.,.)$-cocoercive operator due to Ahmad *et al.* [7], we introduce and study a system of generalized nonlinear quasi-variational-like inclusions and corresponding system of generalized resolvent equations in real Hilbert spaces. Our results can be viewed as an extension and generalization of some known results especially contained in [150, 199, 202, 203].

In Section 5.2, we give the brief introduction of $H(.,.)$-cocoercive operator and its properties investigated by Ahmad *et al.* [7].

In Section 5.3, we consider a system of generalized nonlinear quasi-variational-like inclusions in real Hilbert spaces. By using the resolvent operators associated with $H(.,.)$-cocoercive operators, we show that the approximate solutions obtained by the iterative algorithm converge to the exact solutions of our system of generalized nonlinear quasi-variational-like inclusions.

In Section 5.4, we establish equivalence between system of generalized nonlinear quasi-variational-like inclusions considered in Section 5.3 and system of generalized resolvent equations. This equivalence is used to suggest an iterative algorithm for finding the approximate solution of system of generalized resolvent equations. We also study the convergence of iterative sequences generated by the proposed algorithm.

5.2 Preliminaries

Throughout this chapter, we consider X to be a real Hilbert space endowed with a norm $||.||$ and an inner product $\langle .,. \rangle$, d is the metric induced by the norm $||.||$, 2^X (resp., $CB(X)$) is the family of all the nonempty (resp., closed and bounded) subsets of X and

$\mathcal{D}(.,.)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(P,Q) = \max\{\sup_{x\in P} d(x,Q), \sup_{y\in Q} d(P,y)\}, \quad \text{for all } P,Q \in CB(X),$$

where $d(x,Q) = \inf_{y\in Q}||x-y||$ and $d(P,y) = \inf_{x\in P}||x-y||$.

The following definitions and results will be used to prove the results of this chapter.

Definition 5.2.1. [7] Let $H : X \times X \to X$ and $A, B : X \to X$ be the single-valued mappings. Then

(i) $H(A,.)$ is said to be μ-cocoercive with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), x-y\rangle \geq \mu||Ax - Ay||^2, \ \ \forall \ \ x,y \in X;$$

(ii) $H(.,B)$ is said to be γ-relaxed cocoercive with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u,Bx) - H(u,By), x-y\rangle \geq (-\gamma)||Bx - By||^2, \ \ \forall \ \ x,y \in X;$$

(iii) $H(A,.)$ is said to be r_1-Lipschitz continuous with respect to A, if there exists a constant $r_1 > 0$ such that

$$||H(Ax,.) - H(Ay,.)|| \leq r_1||x-y||, \ \ \forall \ \ x,y \in X;$$

(iv) $H(.,B)$ is said to be s_1-Lipschitz continuous with respect to B, if there exists a constant $s_1 > 0$ such that

$$||H(.,Bx) - H(.,By)|| \leq s_1||x-y||, \ \ \forall \ \ x,y \in X.$$

Definition 5.2.2. [7] Let $H : X \times X \to X$ and $A, B : X \to X$ be the single-valued mappings. Then the set-valued mapping $M : X \to 2^X$ is said to be $H(.,.)$-cocoercive with respect to A and B (or simply $H(.,.)$-cocoercive in the sequel), if

(i) M is cocoercive;

(ii) $(H(A,B) + \rho M)(X) = X$, for every $\rho > 0$.

Example 5.2.1. [7] Let $X = \mathbb{R}^2$ with the usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Ax = (2x_1 - 2x_2, -2x_1 + 2x_2)$$

$$By = (-y_1 + y_2, -y_2), \quad \text{for all } x, y \in \mathbb{R}^2.$$

Suppose that $H(A, B) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$H(Ax, By) = Ax + By, \quad \text{for all } x, y \in \mathbb{R}^2.$$

Then it is easy to check that $H(A, B)$ is $1/6$-cocoercive with respect to A and $1/2$-relaxed cocoercive with respect to B.

Let $M = I$, where I is the identity mapping. Then, M is $H(.,.)$-cocoercive mapping with respect to A and B.

Example 5.2.2. [7] Let $X = \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the space of all 2×2 real symmetric matrices. Let $H(Ax, By) = x^2 - y$, for all $x, y \in \mathbb{S}^2$ and $M = I$. Then for $\rho = 1$, we have

$$\big(H(A, B) + M\big)(x) = x^2 - x + x = x^2,$$

but

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \notin \big(H(A, B) + M\big)(\mathbb{S}^2),$$

because $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not the square of any 2×2 real symmetric matrix. Hence, M is not $H(.,.)$-cocoercive with respect to A and B.

Theorem 5.2.1. [7] Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be an $H(.,.)$-cocoercive operator with respect to A and B. Then the operator $\big(H(A, B) + \rho M\big)^{-1}$ is single-valued.

Definition 5.2.3. [7] Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be an $H(.,.)$-cocoercive operator with respect to A and B. Then the resolvent operator $R_{\rho,M}^{H(.,.)} : X \to X$ is defined by

$$R_{\rho,M}^{H(.,.)}(x) = \big(H(A, B) + \rho M\big)^{-1}(x), \quad \forall \ x \in X.$$

Theorem 5.2.2. [7] Let $H(A, B)$ be μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be an $H(.,.)$-cocoercive operator with respect to A and B. Then the resolvent operator $R_{\rho,M}^{H(.,.)} : X \to X$ is $\frac{1}{\mu\alpha^2 - \gamma\beta^2}$-Lipschitz continuous, *i.e.*,

$$||R_{\rho,M}^{H(.,.)}(x) - R_{\rho,M}^{H(.,.)}(y)|| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}||x - y||, \quad \text{for all } x, y \in X.$$

5.3 System of Generalized Nonlinear Quasi-Variational Like Inclusions

Throughout the rest of the chapter, unless otherwise stated, suppose that for each $i \in \{1,2\}$, X_i be real Hilbert spaces, $CB(X_i)$ the family of all bounded and closed subsets of X_i, $H_i : X_i \times X_i \to X_i$, $A_i, B_i, f_i : X_i \to X_i$, $P_i : X_1 \times X_2 \to X_i$, are single-valued mappings and $S_i : X_i \to \mathrm{CB}(X_i)$ are set-valued mappings. Let $M_1 : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each, $x \in X_1$, $M_1(.,x)$ is an $H_1(.,.)$-cocoercive operator with respect to A_1 and B_1 and $M_2 : X_2 \times X_2 \to 2^{X_2}$ is a set-valued mapping such that, for each, $y \in X_2$, $M_2(.,y)$ is an $H_2(.,.)$-cocoercive operator with respect to A_2 and B_2. Assume that $f_1(X_1) \cap$ dom $(M_1(.,x)) \neq \emptyset$, for each $x \in X_1$ and $f_2(X_2) \cap$ dom $(M_2(.,y)) \neq \emptyset$, for each $y \in X_2$. Then we consider the following system of generalized nonlinear quasi-variational-like inclusions:

Find $(x,y) \in X_1 \times X_2$ with $u \in S_1(x)$, $v \in S_2(y)$ such that,

$$\begin{cases} 0 \in P_1(u,v) + M_1(f_1(x),x), \\ 0 \in P_2(u,v) + M_2(f_2(y),y). \end{cases} \tag{5.3.1}$$

Special cases of system (5.3.1):

(i) If for each $(x,y) \in X_1 \times X_2$, $M_1(f_1(x),x) = M_1(x)$ and $M_2(f_2(y),y) = M_2(y)$, $S_1 = S_2 = I$, then system (5.3.1) reduces to the following system considered by Verma [199]:

$$\begin{cases} 0 \in P_1(x,y) + M_1(x), \\ 0 \in P_2(x,y) + M_2(y). \end{cases} \tag{5.3.2}$$

(ii) If $X_1 = X_2$, $S_1 = S_2 = I$, $f_1 = f_2 = f$, $P_1(u,v) = P_2(u,v) = P(.)$, for each $(x,y) \in X_1 \times X_2$, $M_1(f_1(x),x) = M_2(f_2(y),y) = M(f(x))$, then problem (5.3.1) reduces to the following problem considered by Xia *et al.* [203]:

$$0 \in P(x) + M(f(x)). \tag{5.3.3}$$

For a suitable choice of the mappings $f_1, f_2, P_1, P_2, H_1, H_2, A_1, B_1, A_2, B_2, S_1, S_2$ and the spaces X_1, X_2, a number of known systems of quasi-variational inequalities, systems of variational inequalities, systems of quasi-variational inclusions, and variational inclusions can be obtained as special cases of the generalized nonlinear quasi-variational-like inclusion problem (5.3.1).

Lemma 5.3.1. For each $i \in \{1,2\}$, let X_i be real Hilbert spaces, $A_i, B_i, f_i : X_i \to X_i$, $P_i : X_1 \times X_2 \to X_i$ are single-valued mappings and $S_i : X_i \to \mathrm{CB}(X_i)$ are set-valued mappings. Let $H_i : X_i \times X_i \to X_i$ be a single-valued mapping such that $H_i(A_i, B_i)$ is μ_i-cocoercive with respect to A_i and γ_i-relaxed cocoercive with respect to B_i, A_i is α_i-expansive and B_i is β_i-Lipschitz continuous, $\mu_i > \gamma_i$, $\alpha_i > \beta_i$. Let $M_1 : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each, $x \in X_1$, $M_1(.,x)$ is an $H_1(.,.)$-cocoercive operator with respect to A_1 and B_1 and $M_2 : X_2 \times X_2 \to 2^{X_2}$ is a set-valued mapping such that, for each, $y \in X_2$, $M_2(.,y)$ is an $H_2(.,.)$-cocoercive operator with respect to A_2 and B_2. Assume that $f_1(X_1) \cap \operatorname{dom}(M_1(.,x)) \neq \emptyset$, for each $x \in X_1$ and $f_2(X_2) \cap \operatorname{dom}(M_2(.,y)) \neq \emptyset$, for each $y \in X_2$. Then $(x,y) \in X_1 \times X_2$ with $u \in S_1(x)$ and $v \in S_2(y)$ is a solution of the system (5.3.1), if and only if

$$\begin{cases} f_1(x) = R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big[H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u,v)\big], \\ f_2(y) = R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big[H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u,v)\big], \end{cases} \tag{5.3.4}$$

where $R^{H_1(.,.)}_{\rho_1,M_1(.,x)} = \big(H_1(.,.) + \rho_1 M_1(.,x)\big)^{-1}$, $R^{H_2(.,.)}_{\rho_2,M_2(.,y)} = \big(H_2(.,.) + \rho_2 M_2(.,y)\big)^{-1}$, and $\rho_1, \rho_2 > 0$ are constants.

Proof. By using the definitions of the resolvent operators $R^{H_1(.,.)}_{\rho_1,M_1(.,x)}$ and $R^{H_2(.,.)}_{\rho_2,M_2(.,y)}$, the conclusion follow directly.

The preceding lemma allows us to suggest the following iterative algorithm for system (5.3.1).

Algorithm 5.3.1. For $(x_0, y_0) \in X_1 \times X_2$ with $u_0 \in S_1(x_0)$, $v_0 \in S_2(y_0)$, compute the sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}$, as follows:

$$f_1(x_{n+1}) = R^{H_1(.,.)}_{\rho_1,M_1(.,x_n)}\big[H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - \rho_1 P_1(u_n, v_n)\big], \tag{5.3.5}$$

$$f_2(y_{n+1}) = R^{H_2(.,.)}_{\rho_2,M_2(.,y_n)}\big[H_2\big(A_2(f_2(y_n)), B_2(f_2(y_n))\big) - \rho_2 P_2(u_n, v_n)\big], \tag{5.3.6}$$

$$u_n \in S_1(x_n), \ \|u_n - u_{n+1}\|_1 \leq \big(1 + \frac{1}{n+1}\big)\ \mathcal{D}_1\big(S_1(x_n), S_1(x_{n+1})\big), \tag{5.3.7}$$

$$v_n \in S_2(y_n), \ \|v_n - v_{n+1}\|_2 \leq \big(1 + \frac{1}{n+1}\big)\ \mathcal{D}_2\big(S_2(y_n), S_2(y_{n+1})\big), \tag{5.3.8}$$

where, $\mathcal{D}_i(.,.)$ is the Hausdorff metric on $CB(X_i)$, $n = 0, 1, 2, \ldots$, and $\rho_1,\ \rho_2 > 0$ are two constants.

Definition 5.3.1. For each $i \in \{1,2\}$, let $P_i : X_1 \times X_2 \to X_i$ be a single-valued mapping and $S_i : X_i \to CB(X_i)$ a set-valued mapping. Then P_i is

(i) ε_i-Lipschitz continuous in the first argument with respect to S_1, if there exists a constant $\varepsilon_i > 0$ such that

$$\|P_i(u_1,.) - P_i(u_2,.)\|_i \leq \varepsilon_i\|u_1 - u_2\|_1,$$

for all $u_1 \in S_1(x_1),\ u_2 \in S_1(x_2)$ and $x_1, x_2 \in X_1$.

(ii) ζ_i-Lipschitz continuous in the second argument with respect to S_2, if there exists a constant $\zeta_i > 0$ such that

$$\|P_i(.,v_1) - P_i(.,v_2)\|_i \leq \zeta_i\|v_1 - v_2\|_2,$$

for all $v_1 \in S_2(y_1),\ v_2 \in S_2(y_2)$ and $y_1, y_2 \in X_2$.

Now, we show the existence of solution of system (5.3.1) and analyze the convergence of iterative Algorithm 5.3.1.

Theorem 5.3.1. For each $i \in \{1,2\}$, let X_i be real Hilbert spaces with norm $\|.\|_i$, $CB(X_i)$ the family of all bounded and closed subsets of X_i, $H_i : X_i \times X_i \to X_i$, $A_i, B_i, f_i : X_i \to X_i$, $P_i : X_1 \times X_2 \to X_i$ are single-valued mappings and $S_i : X_i \to CB(X_i)$ are set-valued mappings. Let $M_1 : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each, $x \in X_1$, $M_1(.,x)$ is an $H_1(.,.)$-cocoercive operator with respect to A_1 and B_1 and $M_2 : X_2 \times X_2 \to 2^{X_2}$ is a set-valued mapping such that, for each, $y \in X_2$, $M_2(.,y)$ is an $H_2(.,.)$-cocoercive operator with respect to A_2 and B_2. Assume that $f_1(X_1) \cap$ dom $(M_1(.,x)) \neq \emptyset$, for each $x \in X_1$ and $f_2(X_2) \cap$ dom $(M_2(.,y)) \neq \emptyset$, for each $y \in X_2$. Assume that for each $i \in \{1,2\}$, S_i is $\mathcal{D}_i$-Lipschitz continuous with constant l_{S_i}, $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and s_i-Lipschitz continuous with respect to B_i, A_i is α_i-expansive and B_i is β_i-Lipschitz continuous, f_i is λ_{f_i}-Lipschitz continuous and ξ_i-strongly monotone and P_i is ε_i-Lipschitz continuous in the first argument with respect to S_1 and ζ_i-Lipschitz continuous in the second argument with respect to S_2. If there exist constants $\rho_1, \rho_2 > 0$, such that

$$\begin{cases} 0 < \frac{(r_1+s_1)\lambda_{f_1}+\rho_1\varepsilon_1 l_{S_1}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)} + \frac{\rho_2\zeta_2 l_{S_2}}{\xi_2\left(\mu_2\alpha_2^2-\gamma_2\beta_2^2\right)} < 1, \\ 0 < \frac{(r_2+s_2)\lambda_{f_2}+\rho_2\varepsilon_2 l_{S_1}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)} + \frac{\rho_1\zeta_1 l_{S_2}}{\xi_1\left(\mu_1\alpha_1^2-\gamma_1\beta_1^2\right)} < 1. \end{cases} \tag{5.3.9}$$

Then there exist $(x,y) \in X_1 \times X_2$, $u \in S_1(x)$ and $v \in S_2(y)$ satisfying the system (5.3.1) and the iterative sequences $\{x_n\}$, $\{y_n\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 5.3.1, converge strongly to x, y, u and v, respectively.

Proof. Since for each $i \in \{1,2\}$, S_i is $\mathcal{D}_i$-Lipschitz continuous with constant l_{S_i}, it follows from Algorithm 5.3.1 that

$$\begin{aligned}\|u_n - u_{n+1}\|_1 &\leq \big(1+\frac{1}{n+1}\big)\ \mathcal{D}_1\big(S_1(x_n), S_1(x_{n+1})\big),\\ &\leq \big(1+\frac{1}{n+1}\big) l_{S_1}\|x_n - x_{n+1}\|_1 \end{aligned}\tag{5.3.10}$$

$$\begin{aligned}\|v_n - v_{n+1}\|_2 &\leq \big(1+\frac{1}{n+1}\big)\ \mathcal{D}_2\big(S_2(y_n), S_2(y_{n+1})\big),\\ &\leq \big(1+\frac{1}{n+1}\big) l_{S_2}\|y_n - y_{n+1}\|_2. \end{aligned}\tag{5.3.11}$$

Using the ξ_1-strong monotonicity of f_1, we have

$$\begin{aligned}\|f_1(x_{n+1}) - f_1(x_n)\|_1\|x_{n+1}-x_n\|_1 &\geq \langle f_1(x_{n+1}) - f_1(x_n), x_{n+1}-x_n\rangle_1,\\ &\geq \xi_1\|x_{n+1}-x_n\|_1^2,\end{aligned}$$

which implies that

$$\|x_{n+1}-x_n\|_1 \leq \frac{1}{\xi_1}\|f_1(x_{n+1}) - f_1(x_n)\|_1. \tag{5.3.12}$$

Similarly, we have

$$\|y_{n+1}-y_n\|_2 \leq \frac{1}{\xi_2}\|f_2(y_{n+1}) - f_2(y_n)\|_2. \tag{5.3.13}$$

Now, we estimate $\|f_1(x_{n+1}) - f_1(x_n)\|_1$ by using Algorithm 5.3.1 and the Lipschitz continuity of $R^{H_1(.,.)}_{\rho_1, M_1(.,x)}$ as follows:

$$\begin{aligned}&\|f_1(x_{n+1}) - f_1(x_n)\|_1\\ &= \|R^{H_1(.,.)}_{\rho_1,M_1(.,x_n)}\big[H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - \rho_1 P_1(u_n, v_n)\big]\\ &\quad -R^{H_1(.,.)}_{\rho_1,M_1(.,x_{n-1})}\big[H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1}))\big) - \rho_1 P_1(u_{n-1}, v_{n-1})\big]\|_1\\ &\leq \frac{1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1}))\big)\|_1\\ &\quad +\frac{\rho_1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\|P_1(u_n,v_n) - P_1(u_{n-1},v_{n-1})\|_1.\end{aligned}\tag{5.3.14}$$

Since $H_1(A_1, B_1)$ is r_1-Lipschitz continuous with respect to A_1, s_1-Lipschitz continuous with respect to B_1 and f_1 is λ_{f_1}-Lipschitz continuous, we have

$$\begin{aligned}&\|H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1}))\big)\|_1\\ &\quad\leq \|H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_n))\big)\|_1\\ &\quad\quad+\|H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_n))\big) - H_1\big(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1}))\big)\|_1\\ &\quad\leq (r_1+s_1)\|f_1(x_n) - f_1(x_{n-1})\|_1\end{aligned}$$

$$\leq (r_1+s_1)\lambda_{f_1}\|x_n-x_{n-1}\|_1. \tag{5.3.15}$$

Using the same argument as for (5.3.15), we have

$$\begin{aligned}\|H_2\big(A_2(f_2(y_n)),B_2(f_2(y_n))\big)-H_2\big(A_2(f_2(y_{n-1})),B_2(f_2(y_{n-1}))\big)\|_2\\ \leq (r_2+s_2)\lambda_{f_2}\|y_n-y_{n-1}\|_2.\end{aligned} \tag{5.3.16}$$

Since P_1 is ε_1-Lipschitz continuous in the first argument, ζ_1-Lipschitz continuous in the second argument and using (5.3.10), (5.3.11), we have

$$\begin{aligned}&\|P_1(u_n,v_n)-P_1(u_{n-1},v_{n-1})\|_1\\ &\quad\leq \|P_1(u_n,v_n)-P_1(u_{n-1},v_n)\|_1+\|P_1(u_{n-1},v_n)-P_1(u_{n-1},v_{n-1})\|_1\\ &\quad\leq \varepsilon_1\|u_n-u_{n-1}\|_1+\zeta_1\|v_n-v_{n-1}\|_2\\ &\quad\leq \varepsilon_1 l_{S_1}\big(1+\frac{1}{n}\big)\|x_n-x_{n-1}\|_1+\zeta_1 l_{S_2}\big(1+\frac{1}{n}\big)\|y_n-y_{n-1}\|_2.\end{aligned} \tag{5.3.17}$$

Similarly, we have

$$\begin{aligned}&\|P_2(u_n,v_n)-P_2(u_{n-1},v_{n-1})\|_2\\ &\quad\leq \|P_2(u_n,v_n)-P_2(u_{n-1},v_n)\|_2+\|P_2(u_{n-1},v_n)-P_2(u_{n-1},v_{n-1})\|_2\\ &\quad\leq \varepsilon_2\|u_n-u_{n-1}\|_1+\zeta_2\|v_n-v_{n-1}\|_2\\ &\quad\leq \varepsilon_2 l_{S_1}\big(1+\frac{1}{n}\big)\|x_n-x_{n-1}\|_1+\zeta_2 l_{S_2}\big(1+\frac{1}{n}\big)\|y_n-y_{n-1}\|_2.\end{aligned} \tag{5.3.18}$$

Combining (5.3.15), (5.3.17) with (5.3.14), we obtain

$$\begin{aligned}&\|f_1(x_{n+1})-f_1(x_n)\|_1\\ &\quad\leq \frac{1}{\mu_1\alpha_1^2-\gamma_1\beta_1^2}\big[(r_1+s_1)\lambda_{f_1}\|x_n-x_{n-1}\|_1+\rho_1\varepsilon_1\big(1+\frac{1}{n}\big)l_{S_1}\|x_n-x_{n-1}\|_1\\ &\qquad+\rho_1\zeta_1\big(1+\frac{1}{n}\big)l_{S_2}\|y_n-y_{n-1}\|_2\big],\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x_n\|_1\leq{}&\frac{(r_1+s_1)\lambda_{f_1}+\rho_1\varepsilon_1\big(1+\frac{1}{n}\big)l_{S_1}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)}\|x_n-x_{n-1}\|_1\\ &+\frac{\rho_1\zeta_1\big(1+\frac{1}{n}\big)l_{S_2}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)}\|y_n-y_{n-1}\|_2.\end{aligned}$$

Let

$$\|x_{n+1}-x_n\|_1\leq \theta_{1n}\|x_n-x_{n-1}\|_1+\theta_{2n}\|y_n-y_{n-1}\|_2, \tag{5.3.19}$$

where

$$\theta_{1n}=\frac{(r_1+s_1)\lambda_{f_1}+\rho_1\varepsilon_1\big(1+\frac{1}{n}\big)l_{S_1}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)},$$

$$\theta_{2n} = \frac{\rho_1\zeta_1\big(1+\frac{1}{n}\big)l_{S_2}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)}.$$

Now, we estimate $\|f_2(y_{n+1})-f_2(y_n)\|_2$ by using Algorithm 5.3.1 and the Lipschitz continuity of $R^{H_2(.,.)}_{\rho_2,M_2(.,y)}$ as follows:

$$\begin{aligned}
&\|f_2(y_{n+1})-f_2(y_n)\|_2\\
&\quad= \|R^{H_2(.,.)}_{\rho_2,M_2(.,y_n)}\big[H_2\big(A_2(f_2(y_n)),B_2(f_2(y_n))\big)-\rho_2P_2(u_n,v_n)\big]\\
&\qquad -R^{H_2(.,.)}_{\rho_2,M_2(.,y_{n-1})}\big[H_2\big(A_2(f_2(y_{n-1})),B_2(f_2(y_{n-1}))\big)-\rho_2P_2(u_{n-1},v_{n-1})\big]\|_2\\
&\quad\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\|H_2\big(A_2(f_2(y_n)),B_2(f_2(y_n))\big)-H_2\big(A_2(f_2(y_{n-1})),B_2(f_2(y_{n-1}))\big)\|_2\\
&\qquad +\frac{\rho_2}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\|P_2(u_n,v_n)-P_2(u_{n-1},v_{n-1})\|_2. \qquad (5.3.20)
\end{aligned}$$

Combining (5.3.16) and (5.3.18) with (5.3.20), we obtain

$$\begin{aligned}
&\|f_2(y_{n+1})-f_2(y_n)\|_2\\
&\quad\le \frac{1}{\mu_2\alpha_2^2-\gamma_2\beta_2^2}\big[(r_2+s_2)\lambda_{f_2}\|y_n-y_{n-1}\|_2+\rho_2\varepsilon_2\big(1+\frac{1}{n}\big)l_{S_1}\|x_n-x_{n-1}\|_1\\
&\qquad +\rho_2\zeta_2\big(1+\frac{1}{n}\big)l_{S_2}\|y_n-y_{n-1}\|_2\big],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_{n+1}-y_n\|_2 \le{}& \frac{(r_2+s_2)\lambda_{f_2}+\rho_2\varepsilon_2\big(1+\frac{1}{n}\big)l_{S_1}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)}\|y_n-y_{n-1}\|_2\\
&+\frac{\rho_2\zeta_2\big(1+\frac{1}{n}\big)l_{S_2}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)}\|x_n-x_{n-1}\|_1.
\end{aligned}$$

Let

$$\|y_{n+1}-y_n\|_2 \le \theta_{3n}\|x_n-x_{n-1}\|_1+\theta_{4n}\|y_n-y_{n-1}\|_2, \qquad (5.3.21)$$

where

$$\theta_{3n} = \frac{\rho_2\zeta_2\big(1+\frac{1}{n}\big)l_{S_2}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)},$$

$$\theta_{4n} = \frac{(r_2+s_2)\lambda_{f_2}+\rho_2\varepsilon_2\big(1+\frac{1}{n}\big)l_{S_1}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)}.$$

Adding (5.3.19) and (5.3.21), we get

$$\begin{aligned}
\|x_{n+1}-x_n\|_1+\|y_{n+1}-y_n\|_2 &\le (\theta_{1n}+\theta_{3n})\|x_n-x_{n-1}\|_1+(\theta_{2n}+\theta_{4n})\|y_n-y_{n-1}\|_2,\\
&\le \theta_n\big[\ \|x_n-x_{n-1}\|_1+\|y_n-y_{n-1}\|_2\ \big], \qquad (5.3.22)
\end{aligned}$$

where $\theta_n = \max\big[(\theta_{1n}+\theta_{3n}),(\theta_{2n}+\theta_{4n})\big]$.

Letting $n \to \infty$, we obtain $\theta_n \to \theta$, where

$$\theta = \max\left[(\theta_1+\theta_3),(\theta_2+\theta_4)\right],$$

$$\theta_1 = \frac{(r_1+s_1)\lambda_{f_1}+\rho_1\varepsilon_1 l_{S_1}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)}, \quad \theta_2 = \frac{\rho_1\zeta_1 l_{S_2}}{\xi_1(\mu_1\alpha_1^2-\gamma_1\beta_1^2)},$$

$$\theta_3 = \frac{\rho_2\zeta_2 l_{S_2}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)}, \quad \theta_4 = \frac{(r_2+s_2)\lambda_{f_2}+\rho_2\varepsilon_2 l_{S_1}}{\xi_2(\mu_2\alpha_2^2-\gamma_2\beta_2^2)}.$$

By (5.3.9), $\theta \in (0,1)$, and (5.3.22), $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exists $(x,y) \in X_1 \times X_2$ such that $x_n \to x,\ y_n \to y$, as $n \to \infty$. From (5.3.10) and (5.3.11), $\{u_n\}$, $\{v_n\}$ are also Cauchy sequences. Thus, there exists $(u,v) \in X_1 \times X_2$ such that $u_n \to u,\ v_n \to v$, as $n \to \infty$.

Now, we prove that $u \in S_1(x)$ and $v \in S_2(y)$. In fact, since $u_n \in S_1(x_n)$ and $v_n \in S_2(y_n)$, we have

$$\begin{aligned} d(u,S_1(x)) &\le \|u-u_n\|_1 + d(u_n,S_1(x)) \\ &\le \|u-u_n\|_1 + \mathcal{D}_1(S_1(x_n),S_1(x)) \\ &\le \|u-u_n\|_1 + l_{S_1}\|u_n-u\|_1 \to 0, \text{ as } n \to \infty, \end{aligned}$$

which implies that $d(u,S_1(x)) = 0$. Since $S_1(x) \in CB(X_1)$, it follows that $u \in S_1(x)$. Similarly, we have $v \in S_2(y)$. By Lemma 5.3.1, it follows that $x,\ y,\ u,\ v$ is a solution of system (5.3.1). This completes the proof.

5.4 System of Generalized Resolvent Equations

In connection with the system of generalized nonlinear quasi-variational-like inclusion problem (5.3.1), we consider the following system of generalized resolvent equations:

Find $(x,y) \in X_1 \times X_2$, $u \in S_1(x)$, $v \in S_2(y)$, $w \in X_1$, $z \in X_2$ such that

$$\begin{cases} P_1(u,v) + \rho_1^{-1} J^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w) = 0, \\ \\ P_2(u,v) + \rho_2^{-1} J^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z) = 0. \end{cases} \tag{5.4.1}$$

Where

$$\begin{cases} J^{H_1(.,.)}_{\rho_1,M_1(.,x)} = I - H_1\big[A_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.)\big), B_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.)\big)\big], \\ \\ J^{H_2(.,.)}_{\rho_2,M_2(.,y)} = I - H_2\big[A_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.)\big), B_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.)\big)\big]. \end{cases} \tag{5.4.2}$$

I is the identity operator, $R^{H_1(.,.)}_{\rho_1,M_1(.,x)}$, $R^{H_2(.,.)}_{\rho_2,M_2(.,y)}$ are the resolvent operators associated with M_1 and M_2 and

$$\begin{cases} H_1\big[A_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)\big), B_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)\big)\big] = \big[H_1\big(A_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big), B_1\big(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big)\big)\big](w), \\ H_2\big[A_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)\big), B_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)\big)\big] = \big[H_2\big(A_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big), B_2\big(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big)\big)\big](z). \end{cases} \tag{5.4.3}$$

Lemma 5.4.1. The system of generalized nonlinear quasi-variational-like inclusions (5.3.1) has a solution (x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in S_1(x)$, $v \in S_2(y)$ if and only if system of generalized resolvent equations (5.4.1) has a solution (w, z, x, y, u, v) with $(x, y) \in X_1 \times X_2$, $u \in S_1(x)$, $v \in S_2(y)$, $w \in X_1$, $z \in X_2$, where

$$\begin{cases} f_1(x) = R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w), \\ \\ f_2(y) = R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z), \end{cases} \tag{5.4.4}$$

and

$$\begin{cases} w = H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u, v), \\ \\ z = H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u, v). \end{cases} \tag{5.4.5}$$

Proof. Let (x, y, u, v) be a solution of system of generalized nonlinear quasi-variational-like inclusions (5.3.1), then by Lemma 5.3.1 it satisfies the following equations:

$$f_1(x) = R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big[H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u, v)\big],$$
$$f_2(y) = R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big[H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u, v)\big].$$

Let

$$w = H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u, v),$$
$$z = H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u, v),$$

from (5.4.4), we have

$$w = H_1\big(A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w))\big) - \rho_1 P_1(u, v),$$
$$z = H_2\big(A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z))\big) - \rho_2 P_2(u, v),$$

by using (5.4.3), we have

$$w = H_1\big(A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.))\big)(w) - \rho_1 P_1(u, v),$$
$$z = H_2\big(A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.))\big)(z) - \rho_2 P_2(u, v),$$

which implies that

$$\begin{aligned}
&\big[I - H_1\big(A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.))\big)\big](w) = -\rho_1 P_1(u,v),\\
&\big[I - H_2\big(A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(.))\big)\big](z) = -\rho_2 P_2(u,v),
\end{aligned}$$

i.e.,

$$\begin{cases} P_1(u,v) + \rho_1^{-1} J^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w) = 0,\\ P_2(u,v) + \rho_2^{-1} J^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z) = 0. \end{cases}$$

Thus, (w,z,x,y,u,v) is a solution of system of generalized resolvent equations (5.4.1).

Conversely, let (w,z,x,y,u,v) is a solution of system of generalized resolvent equations (5.4.1), then

$$\begin{aligned}
\rho_1 P_1(u,v) &= -J^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)\\
\rho_2 P_2(u,v) &= -J^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z),
\end{aligned}$$

from (5.4.2), it follows that

$$\begin{aligned}
\rho_1 P_1(u,v) &= -\big[I - H_1\big(A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(.))\big)\big](w)\\
&= H_1\big[A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w))\big] - w\\
w &= H_1\big[A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w))\big] - \rho_1 P_1(u,v)\\
R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w) &= R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big[H_1\big(A_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w)), B_1(R^{H_1(.,.)}_{\rho_1,M_1(.,x)}(w))\big) - \rho_1 P_1(u,v)\big],
\end{aligned}$$

by using (5.4.4), we have

$$f_1(x) = R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big[H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u,v)\big].$$

Again from (5.4.2), we have

$$\begin{aligned}
\rho_2 P_2(u,v) &= -\big[I - H_2\big(A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)})\big)\big](z)\\
&= H_2\big[A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z))\big] - z\\
z &= H_2\big[A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z))\big] - \rho_2 P_2(u,v)\\
R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z) &= R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big[H_2\big(A_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z)), B_2(R^{H_2(.,.)}_{\rho_2,M_2(.,y)}(z))\big) - \rho_2 P_2(u,v)\big],
\end{aligned}$$

by using (5.4.4), we have

$$f_2(y) = R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big[H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u,v)\big].$$

Thus, we have

$$\begin{aligned}
f_1(x) &= R^{H_1(.,.)}_{\rho_1,M_1(.,x)}\big[H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u,v)\big]\\
f_2(y) &= R^{H_2(.,.)}_{\rho_2,M_2(.,y)}\big[H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u,v)\big].
\end{aligned}$$

Hence, by Lemma 5.3.1, (x, y, u, v) is a solution of system of generalized nonlinear quasi-variational-like inclusions (5.3.1).

In view of Lemma 5.3.1 and Lemma 5.4.1, we construct the following iterative algorithm for solving the system of generalized resolvent equations (5.4.1).

Algorithm 5.4.1. For any $(w_0, z_0, x_0, y_0, u_0, v_0)$ with $(x_0, y_0) \in X_1 \times X_2$, $u_0 \in S_1(x_0)$, $v_0 \in S_2(y_0)$, $w_0 \in X_1$, $z_0 \in X_2$, compute the sequences $\{w_n\}, \{z_n\}, \{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ by the following iterative schemes:

$$f_1(x_n) = R^{H_1(.,.)}_{\rho_1, M_1(.,x_n)}(w_n), \tag{5.4.6}$$

$$f_2(y_n) = R^{H_2(.,.)}_{\rho_2, M_2(.,y_n)}(z_n), \tag{5.4.7}$$

$$u_n \in S_1(x_n) : \|u_n - u_{n+1}\|_1 \leq \big(1 + \frac{1}{n+1}\big)\mathcal{D}_1\big(S_1(x_n), S_1(x_{n+1})\big), \tag{5.4.8}$$

$$v_n \in S_2(y_n) : \|v_n - v_{n+1}\|_2 \leq \big(1 + \frac{1}{n+1}\big)\mathcal{D}_2\big(S_2(y_n), S_2(y_{n+1})\big), \tag{5.4.9}$$

$$w_{n+1} = H_1\big(A_1(f_1(x_n)), B_1(f_1(x_n))\big) - \rho_1 P_1(u_n, v_n), \tag{5.4.10}$$

$$z_{n+1} = H_2\big(A_2(f_2(y_n)), B_2(f_2(y_n))\big) - \rho_2 P_2(u_n, v_n), \tag{5.4.11}$$

where, $\mathcal{D}_i(.,.)$ is the Hausdorff metric on $CB(X_i)$, $n = 0, 1, 2, ...$ and $\rho_1, \rho_2 > 0$ are two constants.

Theorem 5.4.1. For each $i \in \{1, 2\}$, let X_i be real Hilbert spaces with norm $\|.\|_i$, $CB(X_i)$ the family of all bounded and closed subsets of X_i, $H_i : X_i \times X_i \to X_i$, $A_i, B_i, f_i : X_i \to X_i$, $P_i : X_1 \times X_2 \to X_i$ are single-valued mappings and $S_i : X_i \to CB(X_i)$ are set-valued mappings. Let $M_1 : X_1 \times X_1 \to 2^{X_1}$ be a set-valued mapping such that, for each, $x \in X_1$, $M_1(.,x)$ is an $H_1(.,.)$-cocoercive operator with respect to A_1 and B_1 and $M_2 : X_2 \times X_2 \to 2^{X_2}$ is a set-valued mapping such that, for each, $y \in X_2$, $M_2(.,y)$ is an $H_2(.,.)$-cocoercive operator with respect to A_2 and B_2. Assume that $f_1(X_1) \cap \text{dom}\,(M_1(.,x)) \neq \emptyset$, for each $x \in X_1$ and $f_2(X_2) \cap \text{dom}\,(M_2(.,y)) \neq \emptyset$, for each $y \in X_2$. Assume that for each $i \in \{1, 2\}$,
(*i*) S_i is $\mathcal{D}_i$-Lipschitz continuous with constant l_{S_i};
(*ii*) $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and s_i-Lipschitz continuous with respect to B_i;
(*iii*) A_i is α_i-expansive and B_i is β_i-Lipschitz continuous;
(*iv*) f_i is λ_{f_i}-Lipschitz continuous and ξ_i-strongly monotone;
(*v*) P_i is ε_i-Lipschitz continuous in the first argument with respect to S_1 and ζ_i-Lipschitz

continuous in the second argument with respect to S_2;
if there exist constants $\rho_1, \rho_2 > 0$, such that

$$\begin{cases} 0 < \frac{(r_1+s_1)\lambda_{f_1}+\rho_1\varepsilon_1 l_{S_1}+\rho_2\varepsilon_2 l_{S_1}}{\left(1-\sqrt{1-2\xi_1+\lambda_{f_1}^2}\right)(\mu_1\alpha_1^2-\gamma_1\beta_1^2)} < 1, \\ 0 < \frac{(r_2+s_2)\lambda_{f_2}+\rho_1\zeta_1 l_{S_2}+\rho_2\zeta_2 l_{S_2}}{\left(1-\sqrt{1-2\xi_2+\lambda_{f_2}^2}\right)(\mu_2\alpha_2^2-\gamma_2\beta_2^2)} < 1. \end{cases} \tag{5.4.12}$$

Then there exist $(x, y) \in X_1 \times X_2$, $w \in X_1, z \in X_2$, $u \in S_1(x)$ and $v \in S_2(y)$ satisfying the system of generalized resolvent equations (5.4.1) and the iterative sequences $\{w_n\}, \{z_n\}, \{x_n\}$, $\{y_n\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 5.4.1 converge strongly to w, z, x, y, u and v, respectively.

Proof. From Algorithm 5.4.1, we have

$$\begin{aligned} \|w_{n+1} - w_n\|_1 &= \|H_1(A_1(f_1(x_n)), B_1(f_1(x_n))) - \rho_1 P_1(u_n, v_n) \\ &\quad -[H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1}))) - \rho_1 P_1(u_{n-1}, v_{n-1})]\|_1 \\ &\leq \|H_1(A_1(f_1(x_n)), B_1(f_1(x_n))) - H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})))\|_1 \\ &\quad +\rho_1\|P_1(u_n, v_n) - P_1(u_{n-1}, v_{n-1})\|_1 \\ &\leq \|H_1(A_1(f_1(x_n)), B_1(f_1(x_n))) - H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_n)))\|_1 \\ &\quad -\|H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_n))) - H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})))\|_1 \\ &\quad +\rho_1\|P_1(u_n, v_n) - P_1(u_{n-1}, v_n)\|_1 + \rho_1\|P_1(u_{n-1}, v_n) - P_1(u_{n-1}, v_{n-1})\|_1. \end{aligned}$$

Since $H_1(A_1, B_1)$ is r_1-Lipschitz continuous with respect to A_1, s_1-Lipschitz continuous with respect to B_1, P_1 is ε_1-Lipschitz continuous in the first argument, ζ_1-Lipschitz continuous in the second argument, f_1 is λ_{f_1}-Lipschitz continuous and S_i is $\mathcal{D}_i$-Lipschitz continuous with constant l_{S_i}, we have

$$\begin{aligned} \|w_{n+1} - w_n\|_1 &\leq \left[(r_1 + s_1)\lambda_{f_1} + \rho_1(1 + \tfrac{1}{n})\varepsilon_1 l_{S_1}\right]\|x_n - x_{n-1}\|_1 \\ &\quad +\rho_1(1 + \frac{1}{n})\zeta_1 l_{S_2}\|y_n - y_{n-1}\|_2. \end{aligned} \tag{5.4.13}$$

Again from Algorithm 5.4.1, we have

$$\begin{aligned} \|z_{n+1} - z_n\|_2 &= \|H_2(A_2(f_2(y_n)), B_2(f_2(y_n))) - \rho_2 P_2(u_n, v_n) \\ &\quad -[H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1}))) - \rho_2 P_2(u_{n-1}, v_{n-1})]\|_2 \\ &\leq \|H_2(A_2(f_2(y_n)), B_2(f_2(y_n))) - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})))\|_2 \\ &\quad +\rho_2\|P_2(u_n, v_n) - P_2(u_{n-1}, v_{n-1})\|_2 \\ &\leq \|H_2(A_2(f_2(y_n)), B_2(f_2(y_n))) - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_n)))\|_2 \\ &\quad -\|H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_n))) - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})))\|_2 \\ &\quad +\rho_2\|P_2(u_n, v_n) - P_2(u_{n-1}, v_n)\|_2 + \rho_2\|P_2(u_{n-1}, v_n) - P_2(u_{n-1}, v_{n-1})\|_2. \end{aligned}$$

Since $H_2(A_2, B_2)$ is r_2-Lipschitz continuous with respect to A_2, s_2-Lipschitz continuous with respect to B_2, P_2 is ε_2-Lipschitz continuous in the first argument, ζ_2-Lipschitz continuous in the second argument, f_2 is λ_{f_2}-Lipschitz continuous and S_i is $\mathcal{D}_i$-Lipschitz continuous with constant l_{S_i}, we have

$$\|z_{n+1} - z_n\|_2 \leq \rho_2\big(1 + \tfrac{1}{n}\big)\varepsilon_2 l_{S_1}\|x_n - x_{n-1}\|_1 + \Big[(r_2 + s_2)\lambda_{f_2} + \rho_2\big(1 + \frac{1}{n}\big)\zeta_2 l_{S_2}\Big]\|y_n - y_{n-1}\|_2. \tag{5.4.14}$$

Adding (5.4.13) and (5.4.14), we get

$$\|w_{n+1} - w_n\|_1 + \|z_{n+1} - z_n\|_2 \leq \big[(r_1 + s_1)\lambda_{f_1} + \rho_1(1 + \tfrac{1}{n})\varepsilon_1 l_{S_1} + \rho_2(1 + \tfrac{1}{n})\varepsilon_2 l_{S_1}\big]\|x_n - x_{n-1}\|_1 + \Big[(r_2 + s_2)\lambda_{f_2} + \rho_1(1 + \frac{1}{n})\zeta_1 l_{S_2} + \rho_2(1 + \frac{1}{n})\zeta_2 l_{S_2}\Big]\|y_n - y_{n-1}\|_2. \tag{5.4.15}$$

By (5.4.6), we have

$$\begin{aligned}\|x_n - x_{n-1}\|_1 &= \|x_n - x_{n-1} - f_1(x_n) + f_1(x_{n-1}) + R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w_n) - R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w_{n-1})\|_1\\ &\leq \|x_n - x_{n-1} - f_1(x_n) + f_1(x_{n-1})\|_1 + \|R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w_n) - R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w_{n-1})\|_1\\ &\leq \|x_n - x_{n-1} - f_1(x_n) + f_1(x_{n-1})\|_1 + \frac{1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\|w_n - w_{n-1}\|_1. \end{aligned} \tag{5.4.16}$$

Since f_1 is ξ_1-strongly monotone and λ_{f_1}-Lipschitz continuous, we have

$$\begin{aligned}&\|x_n - x_{n-1} - f_1(x_n) + f_1(x_{n-1})\|_1^2\\ &= \|x_n - x_{n-1}\|_1^2 - 2\langle f_1(x_n) - f_1(x_{n-1}), x_n - x_{n-1}\rangle_1 + \|f_1(x_n) - f_1(x_{n-1})\|_1^2\\ &\leq \|x_n - x_{n-1}\|_1^2 - 2\xi_1\|x_n - x_{n-1}\|_1^2 + \lambda_{f_1}^2\|x_n - x_{n-1}\|_1^2\\ &= (1 - 2\xi_1 + \lambda_{f_1}^2)\|x_n - x_{n-1}\|_1^2. \end{aligned} \tag{5.4.17}$$

Combining (5.4.16) and (5.4.17), we have

$$\|x_n - x_{n-1}\|_1 \leq \sqrt{1 - 2\xi_1 + \lambda_{f_1}^2}\,\|x_n - x_{n-1}\|_1 + \frac{1}{\mu_1\alpha_1^2 - \gamma_1\beta_1^2}\|w_n - w_{n-1}\|_1,$$

which implies that

$$\|x_n - x_{n-1}\|_1 \leq \frac{1}{\big(1 - \sqrt{1 - 2\xi_1 + \lambda_{f_1}^2}\big)(\mu_1\alpha_1^2 - \gamma_1\beta_1^2)}\|w_n - w_{n-1}\|_1. \tag{5.4.18}$$

By (5.4.7), we have

$$\begin{aligned}\|y_n - y_{n-1}\|_2 &= \|y_n - y_{n-1} - f_2(y_n) + f_2(y_{n-1}) + R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z_n) - R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z_{n-1})\|_2\\ &\leq \|y_n - y_{n-1} - f_2(y_n) + f_2(y_{n-1})\|_2 + \|R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z_n) - R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z_{n-1})\|_2\\ &\leq \|y_n - y_{n-1} - f_2(y_n) + f_2(y_{n-1})\|_2 + \frac{1}{\mu_2\alpha_2^2 - \gamma_2\beta_2^2}\|z_n - z_{n-1}\|_2. \end{aligned} \tag{5.4.19}$$

Since f_2 is ξ_2-strongly monotone and λ_{f_2}-Lipschitz continuous, we have

$$\begin{aligned}
&\|y_n - y_{n-1} - f_2(y_n) + f_2(y_{n-1})\|_2^2 \\
&\quad = \|y_n - y_{n-1}\|_2^2 - 2\langle f_2(y_n) - f_2(y_{n-1}), y_n - y_{n-1}\rangle_2 + \|f_2(y_n) - f_2(y_{n-1})\|_2^2 \\
&\quad \leq \|y_n - y_{n-1}\|_2^2 - 2\xi_2\|y_n - y_{n-1}\|_2^2 + \lambda_{f_2}^2\|y_n - y_{n-1}\|_2^2 \\
&\quad = (1 - 2\xi_2 + \lambda_{f_2}^2)\|y_n - y_{n-1}\|_2^2.
\end{aligned} \tag{5.4.20}$$

Combining (5.4.19) and (5.4.20), we have

$$\|y_n - y_{n-1}\|_2 \leq \sqrt{1 - 2\xi_2 + \lambda_{f_2}^2}\|y_n - y_{n-1}\|_2 + \frac{1}{\mu_2\alpha_2^2 - \gamma_2\beta_2^2}\|z_n - z_{n-1}\|_2,$$

which implies that

$$\|y_n - y_{n-1}\|_2 \leq \frac{1}{\left(1 - \sqrt{1 - 2\xi_2 + \lambda_{f_2}^2}\right)(\mu_2\alpha_2^2 - \gamma_2\beta_2^2)}\|z_n - z_{n-1}\|_2. \tag{5.4.21}$$

Using (5.4.18) and (5.4.21), (5.4.15) becomes

$$\|w_{n+1} - w_n\|_1 + \|z_{n+1} - z_n\|_2 \leq \Theta_{1n}\|w_n - w_{n-1}\|_1 + \Theta_{2n}\|z_n - z_{n-1}\|_2, \tag{5.4.22}$$

where

$$\Theta_{1n} = \frac{(r_1 + s_1)\lambda_{f_1} + \rho_1(1 + \frac{1}{n})\varepsilon_1 l_{S_1} + \rho_2(1 + \frac{1}{n})\varepsilon_2 l_{S_1}}{\left(1 - \sqrt{1 - 2\xi_1 + \lambda_{f_1}^2}\right)(\mu_1\alpha_1^2 - \gamma_1\beta_1^2)}$$

$$\Theta_{2n} = \frac{(r_2 + s_2)\lambda_{f_2} + \rho_1(1 + \frac{1}{n})\zeta_1 l_{S_2} + \rho_2(1 + \frac{1}{n})\zeta_2 l_{S_2}}{\left(1 - \sqrt{1 - 2\xi_2 + \lambda_{f_2}^2}\right)(\mu_2\alpha_2^2 - \gamma_2\beta_2^2)}$$

$$\|w_{n+1} - w_n\|_1 + \|z_{n+1} - z_n\|_2 \leq \Theta_n\left[\|w_n - w_{n-1}\|_1 + \|z_n - z_{n-1}\|_2\right],$$

where $\Theta_n = \max\{\Theta_{1n}, \Theta_{2n}\}$

Letting $n \to \infty$, we obtain $\Theta_n \to \Theta$, where

$$\Theta = \max\{\Theta_1, \Theta_2\},$$

$$\Theta_1 = \frac{(r_1 + s_1)\lambda_{f_1} + \rho_1\varepsilon_1 l_{S_1} + \rho_2\varepsilon_2 l_{S_1}}{\left(1 - \sqrt{1 - 2\xi_1 + \lambda_{f_1}^2}\right)(\mu_1\alpha_1^2 - \gamma_1\beta_1^2)}$$

$$\Theta_2 = \frac{(r_2 + s_2)\lambda_{f_2} + \rho_1\zeta_1 l_{S_2} + \rho_2\zeta_2 l_{S_2}}{\left(1 - \sqrt{1 - 2\xi_2 + \lambda_{f_2}^2}\right)(\mu_2\alpha_2^2 - \gamma_2\beta_2^2)}.$$

By (5.4.12), we know that $0 < \Theta < 1$ and so (5.4.22) implies that $\{w_n\}$ and $\{z_n\}$ are both Cauchy sequences. Thus, there exist $w \in X_1$ and $z \in X_2$ such that $w_n \to w$ and $z_n \to z$, as $n \to \infty$.

From (5.4.18) and (5.4.21) it follows that $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences, *i.e.*, there exist $x \in X_1$ and $y \in X_2$ such that $x_n \to x$ and $y_n \to y$, as $n \to \infty$.
Also, from (5.4.8) and (5.4.9), we have

$$\begin{aligned}
\|u_n - u_{n-1}\|_1 &\leq \left(1+\frac{1}{n}\right)\mathcal{D}_1\big(S_1(x_n), S_1(x_{n-1})\big) \\
&\leq \left(1+\frac{1}{n}\right)l_{S_1}\|x_n - x_{n-1}\|_1 \\
\|v_n - v_{n-1}\|_2 &\leq \left(1+\frac{1}{n}\right)\mathcal{D}_2\big(S_2(y_n), S_2(y_{n-1})\big) \\
&\leq \left(1+\frac{1}{n}\right)l_{S_2}\|y_n - y_{n-1}\|_2,
\end{aligned}$$

and hence $\{u_n\}$ and $\{v_n\}$ are also Cauchy sequences, so there exist $u \in X_1$ and $v \in X_2$ such that $u_n \to u$ and $v_n \to v$, as $n \to \infty$.
Now, we will show that $u \in S_1(x)$ and $v \in S_2(y)$. In fact, since $u_n \in S_1(x_n)$ and $v_n \in S_2(y_n)$, we have

$$\begin{aligned}
d(u, S_1(x)) &\leq \|u - u_n\|_1 + d(u_n, S_1(x)) \\
&\leq \|u - u_n\|_1 + \mathcal{D}_1(S_1(x_n), S_1(x)) \\
&\leq \|u - u_n\|_1 + l_{S_1}\|u - u_n\| \to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies that $d(u, S_1(x)) = 0$. Since $S_1(x) \in CB(X_1)$, it follows that $u \in S_1(x)$. Similarly, we can show that $v \in S_2(y)$. By continuity of $H_1, A_1, B_1, H_2, A_2, B_2, M_1, M_2, S_1, S_2, R^{H_1(.,.)}_{\rho_1, M_1(.,x)}, R^{H_2(.,.)}_{\rho_2, M_2(.,y)}$ and Algorithm 5.4.1, we have

$$\begin{aligned}
w &= H_1\big(A_1(f_1(x)), B_1(f_1(x))\big) - \rho_1 P_1(u, v), \\
&= H_1\big(A_1(R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w)), B_1(R^{H_1(.,.)}_{\rho_1, M_1(.,x)}(w))\big) - \rho_1 P_1(u, v), \\
z &= H_2\big(A_2(f_2(y)), B_2(f_2(y))\big) - \rho_2 P_2(u, v), \\
&= H_2\big(A_2(R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z)), B_2(R^{H_2(.,.)}_{\rho_2, M_2(.,y)}(z))\big) - \rho_2 P_2(u, v).
\end{aligned}$$

By Lemma 5.4.1, the required result follows.

Chapter 6

Generalized $H(.,.,.)$-η-Cocoercive Operators with an Application

6.1 Introduction

Recently, in order to study extensively variational inequalities and variational inclusions, which are providing mathematical models to some problems arising in economics, mechanics, and engineering sciences, Huang and Fang [82] firstly introduced the generalized m-accretive mappings and gave the definition of resolvent operator for the generalized m-accretive mappings in Banach spaces. Also, they have discussed some properties of their resolvent operator. Thereafter, Fang and Huang [62], Lan *et al.* [120], Zhang [208] and others introduced and studied several generalized operators such as H-accretive, A-η-accretive and G-η-monotone mappings etc. For further details on such operators one can consult [83, 60, 61, 63, 199, 200, 201, 208, 86, 85, 84, 121] and references therein.

Recently, Zou and Huang [209] introduced and studied $H(.,.)$-accretive operators, Kazmi *et al.* [108, 109, 110] introduced and studied generalized $H(.,.)$-accretive operators and $H(.,.)$-η-proximal point mappings. Wang *et al.* [202] introduced and studied $H(.,.)$-η-accretive operators. Very recently, Ahmad *et al.* [7] introduced and studied $H(.,.)$-cocoercive operators.

Motivated by the above cited relevant papers, we investigate a notion of generalized accretive mapping known as generalized $H(.,.,.)$-η-cocoercive operator in q-uniformaly smooth Banach spaces and prove that the resolvent operator associated with $H(.,.,.)$-η-cocoercive operator is single-valued and Lipschitz continuous. Some examples are

The contents of this chapter have been communicated in Filomat Journal.

constructed to illustrate the definition of $H(.,.,.)$-η-cocoercive operator. Further, as an application, we consider the system of generalized set-valued mixed-quasi variational like inclusion problem involving generalized $H(.,.,.)$-η-cocoercive operators in q-uniformaly smooth Banach spaces. By using the techniques of resolvent operator, an iterative algorithm for solving the system of set-valued mixed quasi variational-like inclusions is constructed and the convergence of iterative sequences generated by the algorithm is proved. Our results can be viewed as a generalization of some known results given in [44, 61, 63, 121, 158, 201].

In Section 6.2, we introduce a class of generalized accretive mapping known as generalized $H(.,.,.)$-η-cocoercive operator in q-uniformaly smooth Banach spaces. We prove some properties of the resolvent operator associated with $H(.,.,.)$-η-cocoercive operator besides furnishing some illustrative examples.

In Section 6.3, we consider a system of generalized set-valued mixed-quasi variational like inclusion problem involving generalized $H(.,.,.)$-η-cocoercive operators in q-uniformaly smooth Banach spaces. By using the resolvent operator technique, we construct an iterative algorithm for finding the approximate solutions of system of generalized set-valued mixed-quasi variational like inclusion and discuss its convergence analysis.

6.2 Generalized $H(.,.,.)$-η-Cocoercive Operator

In this section, we introduce the notion of generalized $H(.,.,.)$-η-cocoercive operator and discuss some of its properties.

Throughout this chapter, we assume that X is a real q-uniformaly smooth Banach space equipped with norm $\|.\|$.

Definition 6.2.1. Let $A, B, C : X \to X$, $\eta : X \times X \to X$ and $H : X \times X \times X \to X$ be *the single-valued mappings. Then*

(i) $H(A,.,.)$ is said to be μ-η-cocoercive with respect to A, if there exists a constant $\mu > 0$ such that

$$\langle H(Ax, u, u) - H(By, u, u), J_q(\eta(x, y))\rangle \geq \mu\|Ax - Ay\|^q, \quad \forall x, y, u \in X;$$

(ii) $H(., B,.)$ is said to be γ-η-relaxed cocoercive with respect to B, if there exists a constant $\gamma > 0$ such that

$$\langle H(u, Bx, u) - H(u, By, u), J_q(\eta(x, y))\rangle \geq (-\gamma)\|Bx - By\|^q, \quad \forall x, y, u \in X;$$

(iii) $H(.,.,C)$ is said to be δ-η-strongly accretive with respect to C, if there exists a constant $\delta > 0$ such that

$$\langle H(u,u,Cx) - H(u,u,Cy), J_q(\eta(x,y))\rangle \geq \delta\|x-y\|^q, \quad \forall x,y,u \in X;$$

(iv) $H(A,.,.)$ is said to be r_1-Lipschitz continuous with respect to A, if there exists a constant $r_1 > 0$ such that

$$\|H(Ax,u,u) - H(Ay,u,u)\| \leq r_1\|x-y\|, \quad \forall x,y,u \in X;$$

(v) $H(.,B,.)$ is said to be s_1-Lipschitz continuous with respect to B, if there exists a constant $s_1 > 0$ such that

$$\|H(u,Bx,u) - H(u,By,u)\| \leq s_1\|x-y\|, \quad \forall x,y,u \in X;$$

(vi) $H(.,.,C)$ is said to be t_1-Lipschitz continuous with respect to C, if there exists a constant $t_1 > 0$ such that

$$\|H(u,u,Cx) - H(u,u,Cy)\| \leq t_1\|x-y\|, \quad \forall x,y,u \in X.$$

Definition 6.2.2. Let $H : X \times X \times X \to X$, $A, B, C : X \to X$ and $\eta : X \times X \to X$ be the single-valued mappings. Let $H(A,B,C)$ be μ-η-cocoercive with respect to A, γ-η-relaxed cocoercive with respect to B, δ-η-strongly accretive with respect to C. Then the set-valued mapping $M : X \to 2^X$ is said to be generalized $H(.,.,.)$-η-cocoercive with respect to the mappings A, B and C, if
(i) M is m-η-relaxed accretive;
(ii) $(H(A,B,C) + \rho M)(X) = X$, for all $\rho > 0$.

Remark 6.2.1.
(i) If $H(A,B,C) = H(A,B)$, A is α-strongly accretive and B is β-relaxed accretive, then generalized $H(.,.,.)$-η-cocoercive operator reduces to $H(.,.)$-η-accretive operator introduced and studied by Wang and Ding [202].
(ii) If we take $H(A,B,C) = H(A,B)$ and $H(A,B)$ be μ-cocoercive with respect to A, γ-relaxed accretive with respect to B, then Definition 6.2.2 reduces to the Definition of $H(.,.)$-mixed mapping introduced by Husain et al. [86].

Example 6.2.1. Let us consider the 2-uniformaly smooth Banach space $X = \mathbb{R}^2$ with the usual inner product. Let $A, B, C : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} Ax &= (\frac{1}{2}x_1 - \frac{1}{2}x_2, -\frac{1}{2}x_1 + x_2), \\ By &= (-\frac{1}{2}y_1 - \frac{1}{2}y_2, \frac{1}{2}y_1 - \frac{1}{2}y_2), \\ Cz &= (\frac{1}{2}z_1 - \frac{1}{4}z_2, \frac{1}{4}z_1 + \frac{1}{3}z_2), \end{aligned}$$

for all $x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2) \in \mathbb{R}^2$.
Suppose that $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} H(Ax, By, Cz) &= Ax + By + Cz, \\ \eta(x, y) &= x - y, \qquad\qquad \forall x, y, z \in \mathbb{R}^2. \end{aligned}$$

Then it is easy to see that H is $\frac{2}{3}$-η-cocoercive with respect to A, 1-η-relaxed cocoercive with respect to B and $\frac{1}{3}$-η-strongly accretive with respect to C.

Example 6.2.2. Let $X, A, B, C, H(A, B, C)$ and η are defined as in Example 6.2.1. Suppose that $M : X \to 2^X$ be defined by $M(x) = (-3\pi, -3x_2)$, $\forall\, x = (x_1, x_2) \in \mathbb{R}^2$.

Then it is easy to show that M is 3-η-relaxed accretive mapping. Furthermore, M is also an $H(.,.,.)$-η-cocoercive operator since $(H(A, B, C) + \rho M)(X) = X$, $\forall\, \rho > 0$.

Theorem 6.2.1. Let set-valued mapping $M : X \to 2^X$ be a generalized $H(.,.,.)$-η-cocoercive operator with respect to the mappings A, B and C. If A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^q - \gamma\beta^q + \delta > m$, then if the following inequality:

$$\langle u - v,\ J_q(\eta(x, y))\rangle \ \geq\ 0, \tag{6.2.1}$$

holds for all $(y, v) \in \text{Graph}(M)$, then $u \in Mx$, where

$$\text{Graph}(M) = \{(a, b) \in X \times X : b \in M(a)\}. \tag{6.2.2}$$

Proof. Suppose on contrary that there exists $(x_0, u_0) \notin \text{Graph}(M)$ such that

$$\langle u_0 - v, J_q(\eta(x_0, y))\rangle \ \geq\ 0, \forall\ (y, v) \in \text{Graph}(M). \tag{6.2.3}$$

Since M is generalized $H(.,.,.)$-η-cocoercive, we know that $(H(A, B, C) + \rho\, M)(X) = X$, holds for all $\rho > 0$, and so there exists $(x_1, u_1) \in \text{Graph}(M)$ such that

$$H(Ax_0, Bx_0, Cx_0) + \rho u_0 = H(Ax_1, Bx_1, Cx_1) + \rho u_1 \in X. \tag{6.2.4}$$

Now,

$$
\begin{aligned}
\rho u_0 - \rho u_1 &= H(Ax_1, Bx_1, Cx_1) - H(Ax_0, Bx_0, Cx_0) \in X,\\
\langle \rho u_0 - \rho u_1, J_q(\eta(x_0, x_1))\rangle &\\
&= -\langle H(Ax_0, Bx_0, Cx_0) - H(Ax_1, Bx_1, Cx_1), J_q(\eta(x_0, x_1))\rangle. \qquad (6.2.5)
\end{aligned}
$$

Setting $(y, v) = (x_1, u_1)$ in (6.2.3) and then from the resultant, (6.2.4) and m-η-relaxed accretivity of M, we obtain

$$
\begin{aligned}
-m\|x_0 - x_1\|^q \le \rho\langle u_0 - u_1, J_q(\eta(x_0, x_1))\rangle &\\
= &- \langle H(Ax_0, Bx_0, Cx_0) - H(Ax_1, Bx_1, Cx_1), J_q(\eta(x_0, x_1))\rangle\\
= &- \langle H(Ax_0, Bx_0, Cx_0) - H(Ax_1, Bx_0, Cx_0), J_q(\eta(x_0, x_1))\rangle\\
&- \langle H(Ax_1, Bx_0, Cx_0) - H(Ax_1, Bx_1, Cx_0), J_q(\eta(x_0, x_1))\rangle\\
&- \langle H(Ax_1, Bx_1, Cx_0) - H(Ax_1, Bx_1, Cx_1), J_q(\eta(x_0, x_1))\rangle. \quad (6.2.6)
\end{aligned}
$$

Since $H(A, B, C)$ is μ-η-cocoercive with respect to A, γ-η-relaxed cocoercive with respect to B, δ-η-strongly accretive with respect to C, A is α-expansive and B is β-Lipschitz continuous, thus (6.2.6) becomes

$$
\begin{aligned}
-m\|x_0 - x_1\|^q \le & -\mu\|Ax_0 - Ax_1\|^q + \gamma\|Bx_0 - Bx_1\|^q - \delta\|x_0 - x_1\|^q\\
\le & -(\mu\alpha^q - \gamma\beta^q + \delta)\|x_0 - x_1\|^q\\
= & -r\|x_0 - x_1\|^q \le 0, \quad \text{where } r = \mu\alpha^q - \gamma\beta^q + \delta\\
\le & -(r - m)\ \|x_0 - x_1\|^q \le 0,
\end{aligned}
$$

which gives $x_0 = x_1$, since $r > m$. By (6.2.3), we have $u_0 = u_1$, a contradiction.
Hence $(x_0, u_0) = (x_1, u_1) \in$ Graph (M) and so $x_0 \in Mu_0$.
This complete the proof.

Theorem 6.2.2. Let set-valued mapping $M : X \to 2^X$ be a generalized $H(.,.,.)$-η-cocoercive operator with respect to the mappings A, B and C. If A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r = \mu\alpha^q - \gamma\beta^q + \delta > \rho m$, then $(H(A, B, C) + \rho M)^{-1}$ is single-valued.

Proof. For any given $x \in X$, let $u, v \in (H(A, B, C) + \rho M)^{-1}(x)$. It follows that

$$
\begin{cases}
-H(Au, Bu, Cu) + x \in \rho Mu,\\
-H(Av, Bv, Cv) + x \in \rho Mv.
\end{cases}
$$

Since M is m-η-relaxed accretive, we have

$$
-m\|u - v\|^q \le \frac{1}{\rho}\langle -H(Au, Bu, Cu) + x - (-H(Av, Bv, Cv) + x), J_q(\eta(u, v))\rangle
$$

$$
\begin{aligned}
-\rho m\|u-v\|^q &\leq \langle -H(Au,Bu,Cu)+x-(-H(Av,Bv,Cv)+x), J_q(\eta(u,v))\rangle \\
&= -\langle H(Au,Bu,Cu)-H(Av,Bv,Cv), J_q(\eta(u,v))\rangle \\
&= -\langle H(Au,Bu,Cu)-H(Av,Bu,Cu), J_q(\eta(u,v))\rangle \\
&\quad -\langle H(Av,Bu,Cu)-H(Av,Bv,Cu), J_q(\eta(u,v))\rangle \\
&\quad -\langle H(Av,Bv,Cu)-H(Av,Bv,Cv), J_q(\eta(u,v))\rangle. \qquad (6.2.7)
\end{aligned}
$$

Since $H(A,B,C)$ is μ-η-cocoercive with respect to A, γ-η-relaxed cocoercive with respect to B and δ-η-strongly accretive with respect to C, A is α-expansive and B is β-Lipschitz continuous, thus (6.2.7) becomes

$$
\begin{aligned}
-\rho m\|u-v\|^q \leq & -\mu\|Au-Av\|^q+\gamma\|Bu-Bv\|^q-\delta\|u-v\|^q \\
\leq & -(\mu\alpha^q-\gamma\beta^q+\delta)\|u-v\|^q \\
= & -r\|u-v\|^q \leq 0, \quad \text{where } r=\mu\alpha^q-\gamma\beta^q+\delta \\
\leq & -(r-\rho m)\ \|u-v\|^q \leq 0,
\end{aligned}
$$

since $r > \rho m$. Hence, it follows that $\|u-v\| \leq 0$. This implies that $u=v$, and so $(H(A,B,C)+\rho M)^{-1}$ is single-valued.

Definition 6.2.3. Let set-valued mapping $M : X \to 2^X$ be a generalized $H(.,.,.)$-η-cocoercive operator with respect to the mappings A, B and C. If A is α-expansive, B is β-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r=\mu\alpha^q-\gamma\beta^q+\delta > \rho m$, then the resolvent operator $R_{\rho,M}^{H(.,.,.)-\eta} : X \to X$ is defined by

$$
R_{\rho,M}^{H(.,.,.)-\eta}(u) = (H(A,B,C)+\rho M)^{-1}(u), \quad \forall\, u \in X. \qquad (6.2.8)
$$

Now we prove that the resolvent operator defined by (6.2.8) is Lipschitz continuous.

Theorem 6.2.3. Let set-valued mapping $M : X \to 2^X$ be a generalized $H(.,.,.)$-η-cocoercive operator with respect to the mappings A, B and C. If A is α-expansive, B is β-Lipschitz continuous, η is τ-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$ with $r=\mu\alpha^q-\gamma\beta^q+\delta > \rho m$, then the resolvent operator $R_{\rho,M}^{H(.,.,.)-\eta} : X \to X$ is $\frac{\tau^{q-1}}{r-\rho m}$-Lipschitz continuous, that is,

$$
\|R_{\rho,M}^{H(.,.,.)-\eta}(u)-R_{\rho,M}^{H(.,.,.)-\eta}(v)\| \leq \frac{\tau^{q-1}}{r-\rho m}\|u-v\|, \quad \forall\, u,v \in X. \qquad (6.2.9)
$$

Proof. Let $u,\ v \in X$ be any given points, It follows from (6.2.8) that

$$
\begin{aligned}
R_{\rho,M}^{H(.,.,.)-\eta}(u) &= (H(A,B,C)+\rho M)^{-1}(u), \\
R_{\rho,M}^{H(.,.,.)-\eta}(v) &= (H(A,B,C)+\rho M)^{-1}(v).
\end{aligned}
$$

$$\frac{1}{\rho}\big(u - H(A(R_{\rho,M}^{H(.,.,.)-\eta}(u)), B(R_{\rho,M}^{H(.,.,.)-\eta}(u)), C(R_{\rho,M}^{H(.,.,.)-\eta}(u)))\big) \in M(R_{\rho,M}^{H(.,.,.)-\eta}(u)),$$
$$\frac{1}{\rho}\big(v - H(A(R_{\rho,M}^{H(.,.,.)-\eta}(v)), B(R_{\rho,M}^{H(.,.,.)-\eta}(v)), C(R_{\rho,M}^{H(.,.,.)-\eta}(v)))\big) \in M(R_{\rho,M}^{H(.,.,.)-\eta}(v)).$$

Let $z_1 = R_{\rho,M}^{H(.,.,.)-\eta}(u)$ and $z_2 = R_{\rho,M}^{H(.,.,.)-\eta}(v)$.

Since M is m-η-relaxed accretive, we have

$$\begin{aligned}\frac{1}{\rho}\langle(u - H(A(z_1), B(z_1), C(z_1))) - (v - H(A(z_2), B(z_2), C(z_2))),\ J_q(\eta(z_1, z_2))\rangle\\ \geq -m\|z_1 - z_2\|^q,\\ \langle u - v - (H(A(z_1), B(z_1), C(z_1)) - H(A(z_2), B(z_2), C(z_2))),\ J_q(\eta(z_1, z_2))\rangle\\ \geq -\rho m\|z_1 - z_2\|^q,\end{aligned}$$

which implies that

$$\begin{aligned}\langle u - v, J_q(\eta(z_1, z_2))\rangle \geq \langle H(A(z_1), B(z_1), C(z_1)) - H(A(z_2), B(z_2), C(z_2)), J_q(\eta(z_1, z_2))\rangle\\ -\rho m\ \|z_1 - z_2\|^q.\end{aligned}$$

Further, we have

$$\begin{aligned}&\|u - v\|\ \tau^{q-1}\|z_1 - z_2\|^{q-1}\\ &\geq \|u - v\|\ \|\eta(z_1, z_2)\|^{q-1}\\ &\geq \langle u - v,\ J_q(\eta(z_1, z_2))\rangle\\ &\geq \langle H(A(z_1), B(z_1), C(z_1)) - H(A(z_2), B(z_2), C(z_2)),\ J_q(\eta(z_1, z_2))\rangle - \rho m\|z_1 - z_2\|^q\\ &\geq \langle H(A(z_1), B(z_1), C(z_1)) - H(A(z_2), B(z_1), C(z_1)), J_q(\eta(z_1, z_2))\rangle\\ &\quad -\langle H(A(z_2), B(z_1), C(z_1)) - H(A(z_2), B(z_2), C(z_1)), J_q(\eta(z_1, z_2))\rangle\\ &\quad -\langle H(A(z_2), B(z_2), C(z_1)) - H(A(z_2), B(z_2), C(z_2)), J_q(\eta(z_1, z_2))\rangle - \rho m\|z_1 - z_2\|^q\\ &\geq \mu\|A(z_1) - A(z_2)\|^q - \gamma\|B(z_1) - B(z_2)\|^q + \delta\|C(z_1) - C(z_2)\|^q - \rho m\|z_1 - z_2\|^q\\ &\geq \mu\alpha^q\|z_1 - z_2\|^q - \gamma\beta^q\|z_1 - z_2\|^q + \delta\|z_1 - z_2\|^q - \rho m\|z_1 - z_2\|^q\\ &= (\mu\alpha^q - \gamma\beta^q + \delta - \rho m)\ \|z_1 - z_2\|^q,\\ &= (r - \rho m)\ \|z_1 - z_2\|^q, \quad \text{where } r = \mu\alpha^q - \gamma\beta^q + \delta,\end{aligned}$$

and hence,

$$\|u - v\|\ \tau^{q-1}\|z_1 - z_2\|^{q-1} \ \geq\ (r - \rho m)\ \|z_1 - z_2\|^q,$$

i.e.,

$$\|R_{\rho,M}^{H(.,.,.)-\eta}(u) - R_{\rho,M}^{H(.,.,.)-\eta}(v)\| \ \leq\ \frac{\tau^{q-1}}{r - \rho m}\ \|u - v\|, \quad \forall\, u, v \in X.$$

This completes the proof.

6.3 An Application of Generalized $H(.,.,.)$-η-Cocoercive Operator

In this section, we formulate a system of generalized set-valued mixed quasi-variational-like inclusions involving $H(.,.,.)$-η-cocoercive operators in q-uniformaly smooth Banach spaces.

Throughout the rest of the paper, unless otherwise stated, suppose that for each $i \in \{1,2\}$, X_i be q_i-uniformaly smooth Banach spaces with norm $\|.\|_i$, $CB(X_i)$ the family of all bounded and closed subsets of X_i, $H_i : X_i \times X_i \times X_i \to X_i$, $\eta_i : X_i \times X_i \to X_i$, $A_i, B_i, C_i, f_i : X_i \to X_i$, $N_i, P_i : X_1 \times X_2 \to X_i$, are single-valued mappings and $S_i : X_1 \to \mathrm{CB}(X_1)$, $T_i : X_2 \to \mathrm{CB}(X_2)$ are set-valued mappings. For each $i \in \{1,2\}$, let $M_i : X_i \to 2^{X_i}$ be a set-valued mapping such that M_i is a generalized $H_i(A_i, B_i, C_i)$-η_i-cocoercive operator with respect to A_i, B_i, C_i and range$(f_i) \cap$ dom$(M_i) \neq \emptyset$. Then we consider the problem of finding (x, y, u, v, w, z) such that $(x,y) \in X_1 \times X_2, u \in S_1(x), v \in T_1(y), w \in S_2(x), z \in T_2(y)$ and

$$\begin{cases} 0 \in N_1(x,y) + P_1(u,v) + M_1(f_1(x)), \\ 0 \in N_2(x,y) + P_2(w,z) + M_2(f_2(y)). \end{cases} \tag{6.3.1}$$

System (6.3.1) is called the system of generalized set-valued mixed quasi-variational-like inclusions.

Special cases of the system (6.3.1):

(i) If for each $i \in \{1,2\}$, $X_i = H_i$ is a real Hilbert space, $f_i \equiv I_i$ (the identity mapping on H_i), $P_1 \equiv 0$, $P_2 \equiv 0$, then the system (6.3.1) reduces to the following system of variational inclusions with (H,η)-monotone operators considered by Fang *et al.* [63], which is to find $(x,y) \in H_1 \times H_2$ such that

$$\begin{cases} 0 \in N_1(x,y) + M_1(x), \\ 0 \in N_2(x,y) + M_2(y). \end{cases} \tag{6.3.2}$$

(ii) If for each $i \in \{1,2\}$, $X_i = H_i$ is a real Hilbert space, $N_1 = N_2 \equiv 0$ and $S_1 \equiv g_1$, $T_2 \equiv g_2$ are two single-valued mappings, then the system (6.3.1) reduces to the following system of generalized nonlinear multi-valued variational inclusions with A-monotone mappings considered by Lan *et al.* [121], which is to find $(x,y) \in H_1 \times H_2, v \in$

$T_1(y), w \in S_2(x)$ such that

$$\begin{cases} 0 \in P_1(g_1(x), v) + M_1(f_1(x)), \\ 0 \in P_2(w, g_2(y)) + M_2(f_2(y)). \end{cases} \tag{6.3.3}$$

If for each $i \in \{1,2\}$, $X_i = H_i$ is a real Hilbert space and M_i is an (H_i, η_i)-monotone operator, then the system (6.3.1) reduces to a system of generalized mixed quasi-variational inclusions with (H_i, η_i)-monotone operators introduced by Peng and Zhu [158], which includes those mathematical models in references [44, 61, 63, 201] as special cases.

Lemma 6.3.1. An element (x, y, u, v, w, z), where $(x, y) \in X_1 \times X_2, u \in S_1(x), v \in T_1(y), w \in S_2(x), z \in T_2(y)$ is a solution of the system (6.3.1), if and only if (x, y, u, v, w, z) satisfies the following relation:

$$\begin{cases} f_1(x) = R^{H_1(.,.,.)-\eta_1}_{\rho_1, M_1}[H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1(N_1(x,y) + P_1(u,v))], \\ f_2(y) = R^{H_2(.,.,.)-\eta_2}_{\rho_2, M_2}[H_2(A_2(f_2(y)), B_2(f_2(y)), C_2(f_2(y))) - \rho_2(N_2(x,y) + P_2(w,z))], \end{cases}$$

where $R^{H_1(.,.,.)-\eta_1}_{\rho_1, M_1} = (H_1(A_1, B_1, C_1) + \rho_1 M_1)^{-1}$, $R^{H_2(.,.,.)-\eta_2}_{\rho_2, M_2} = (H_2(A_2, B_2, C_2) + \rho_2 M_2)^{-1}$, and $\rho_1, \rho_2 > 0$ are two constants.

Proof. Consider first that an element (x, y, u, v, w, z) is a solution to system (6.3.1), then it follows that

$$\begin{aligned}
&0 \in N_1(x,y) + P_1(u,v) + M_1(f_1(x)), \\
&\Longrightarrow H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \in H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \\
&\qquad\qquad + \rho_1 N_1(x,y) + \rho_1 P_1(u,v) + \rho_1 M_1(f_1(x)) \\
&\Longrightarrow H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1 N_1(x,y) - \rho_1 P_1(u,v) \\
&\qquad\qquad \in (H_1(A_1, B_1, C_1) + \rho_1 M_1)(f_1(x)) \\
&\Longrightarrow f_1(x) = (H_1(A_1, B_1, C_1) + \rho_1 M_1)^{-1}[H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \\
&\qquad\qquad - \rho_1 N_1(x,y) - \rho_1 P_1(u,v)] \\
&\Longrightarrow f_1(x) = R^{H_1(.,.,.)-\eta_1}_{\rho_1, M_1}[H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1 N_1(x,y) - \rho_1 P_1(u,v)].
\end{aligned}$$

In a similar way, we can show that

$$f_2(y) = R^{H_2(.,.,.)-\eta_2}_{\rho_2, M_2}[H_2(A_2(f_2(y)), B_2(f_2(y)), C_2(f_2(y))) + \rho_2 N_2(x,y) + \rho_2 P_2(w,z)].$$

A similar proof follows for the converse part:

$$\begin{aligned}
& f_1(x) = R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}[H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1 N_1(x,y) - \rho_1 P_1(u,v)], \\
& \Longrightarrow f_1(x) = (H_1(A_1,B_1,C_1) + \rho_1 M_1)^{-1}[H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \\
& \qquad\qquad -\rho_1 N_1(x,y) - \rho_1 P_1(u,v)] \\
& \Longrightarrow H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1 N_1(x,y) - \rho_1 P_1(u,v) \\
& \qquad\qquad \in (H_1(A_1,B_1,C_1) + \rho_1 M_1)(f_1(x)) \\
& \Longrightarrow H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \in H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \\
& \qquad\qquad +\rho_1 N_1(x,y) + \rho_1 P_1(u,v) + \rho_1 M_1(f_1(x)) \\
& \Longrightarrow 0 \in \ N_1(x,y) + P_1(u,v) + M_1(f_1(x)).
\end{aligned}$$

In a similar way, we can show that

$$0 \in \ N_2(x,y) + P_2(w,z) + M_2(f_2(y)).$$

Algorithm 6.3.1. For each $x, y \in X_1 \times X_2$, $F_1(x) \subseteq f_1(X_1)$, $F_2(y) \subseteq f_2(X_2)$, where $F_1 : X_1 \to 2^{X_1}$ and $F_2 : X_2 \to 2^{X_2}$ be the set-valued mappings defined by

$$F_1(x) = \bigcup_{u\in S_1(x)} \bigcup_{v \in T_1(y)} \Big(R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}(H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) \\ -\rho_1 N_1(x,y) - \rho_1 P_1(u,v))\Big), \tag{6.3.4}$$

$$F_2(y) = \bigcup_{w\in S_2(x)} \bigcup_{z \in T_2(y)} \Big(R^{H_2(.,.,.)-\eta_2}_{\rho_2,M_2}(H_2(A_2(f_2(y)), B_2(f_2(y)), C_2(f_2(y))) \\ -\rho_2 N_2(x,y) - \rho_2 P_2(w,z))\Big), \tag{6.3.5}$$

where $M_1 : X_1 \to 2^{X_1}$ and $M_2 : X_2 \to 2^{X_2}$ be the set-valued mappings such that M_1 is generalized $H_1(A_1, B_1, C_1)$-η_1-cocoercive operator with respect to A_1, B_1, C_1 and M_2 is generalized $H_2(A_2, B_2, C_2)$-η_2-cocoercive operator with respect to A_2, B_2, C_2.

For given $(x_0, y_0) \in X_1 \times X_2, u_0 \in S_1(x_0), v_0 \in T_1(y_0), w_0 \in S_2(x_0), z_0 \in T_2(y_0)$, let

$$\begin{aligned}
a_0 = R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}&(H_1(A_1(f_1(x_0)), B_1(f_1(x_0)), C_1(f_1(x_0))) \\
&-\rho_1 N_1(x_0,y_0) - \rho_1 P_1(u_0,v_0)) \in F_1(x_0) \subseteq f_1(X_1),
\end{aligned}$$

$$\begin{aligned}
b_0 = R^{H_2(.,.,.)-\eta_2}_{\rho_2,M_2}&(H_2(A_2(f_2(y_0)), B_2(f_2(y_0)), C_2(f_2(y_0))) \\
&-\rho_2 N_2(x_0,y_0) - \rho_2 P_2(w_0,z_0)) \in F_2(y_0) \subseteq f_2(X_2).
\end{aligned}$$

Hence, there exists $(x_1, y_1) \in X_1 \times X_2$ such that $a_0 = f_1(x_1)$, $b_0 = f_2(y_1)$. Since $u_0 \in S_1(x_0) \in CB(X_1)$, $v_0 \in T_1(y_0) \in CB(X_2)$, $w_0 \in S_2(x_0) \in CB(X_1)$ and $z_0 \in T_2(y_0) \in CB(X_2)$, then by Nadler's result [138], there exist $u_1 \in S_1(x_1)$, $v_1 \in T_1(y_1)$, $w_1 \in S_2(x_1)$ and $z_1 \in T_2(y_1)$ such that

$$\begin{aligned}
\|u_0 - u_1\|_1 &\leq (1 + 1^{-1})\ \mathcal{D}_1(S_1(x_0), S_1(x_1)),\\
\|v_0 - v_1\|_2 &\leq (1 + 1^{-1})\ \mathcal{D}_2(T_1(y_0), T_1(y_1)),\\
\|w_0 - w_1\|_1 &\leq (1 + 1^{-1})\ \mathcal{D}_1(S_2(x_0), S_2(x_1)),\\
\|z_0 - z_1\|_2 &\leq (1 + 1^{-1})\ \mathcal{D}_2(T_2(y_0), T_2(y_1)),
\end{aligned}$$

where $\mathcal{D}_i(.,.)$ is the Hausdorff metric on $CB(X_i)$, for $i = 1, 2$.
Let

$$\begin{aligned}
a_1 = R_{\rho_1, M_1}^{H_1(.,.,.)-\eta_1}&(H_1(A_1(f_1(x_1)), B_1(f_1(x_1)), C_1(f_1(x_1)))\\
&-\rho_1 N_1(x_1, y_1) - \rho_1 P_1(u_1, v_1)) \in F_1(x_1) \subseteq f_1(X_1),
\end{aligned}$$

$$\begin{aligned}
b_1 = R_{\rho_2, M_2}^{H_2(.,.,.)-\eta_2}&(H_2(A_2(f_2(y_1)), B_2(f_2(y_1)), C_2(f_2(y_1)))\\
&-\rho_2 N_2(x_1, y_1) - \rho_2 P_2(w_1, z_1)) \in F_2(y_1) \subseteq f_2(X_2).
\end{aligned}$$

Hence, there exists $(x_2, y_2) \in X_1 \times X_2$ such that $a_1 = f_1(x_2)$, $b_1 = f_2(y_2)$. By induction, we can define iterative sequences $\{x_n\}$, $\{f_1(x_n)\}$, $\{y_n\}$, $\{f_2(y_n)\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{z_n\}$ as follows:

$$\begin{aligned}
f_1(x_{n+1}) = R_{\rho_1, M_1}^{H_1(.,.,.)-\eta_1}&(H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
&-\rho_1 N_1(x_n, y_n) - \rho_1 P_1(u_n, v_n)),
\end{aligned} \tag{6.3.6}$$

$$\begin{aligned}
f_2(y_{n+1}) = R_{\rho_2, M_2}^{H_2(.,.,.)-\eta_2}&(H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n)))\\
&-\rho_2 N_2(x_n, y_n) - \rho_2 P_2(w_n, z_n)),
\end{aligned} \tag{6.3.7}$$

$$u_n \in S_1(x_n), \quad \|u_n - u_{n+1}\|_1 \leq \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}_1(S_1(x_n), S_1(x_{n+1})), \tag{6.3.8}$$

$$v_n \in T_1(y_n), \quad \|v_n - v_{n+1}\|_2 \leq \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}_2(T_1(y_n), T_1(y_{n+1})), \tag{6.3.9}$$

$$w_n \in S_2(x_n), \quad \|w_n - w_{n+1}\|_1 \leq \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}_1(S_2(x_n), S_2(x_{n+1})), \tag{6.3.10}$$

$$z_n \in T_2(y_n), \quad \|z_n - z_{n+1}\|_2 \leq \left(1 + \frac{1}{n+1}\right)\ \mathcal{D}_2(T_2(y_n), T_2(y_{n+1})), \tag{6.3.11}$$

for all $n = 0, 1, 2, \ldots$, and $\rho_1, \rho_2 > 0$ are constants.

Definition 6.3.1. For $i \in \{1, 2\}$, let $N_i : X_1 \times X_2 \to X_i$ be a single-valued mapping. Then N_i is

(i) ζ_i-Lipschitz continuous in the first argument, if there exists a constant $\zeta_i > 0$ such that

$$\|N_i(x_1, .) - N_i(x_2, .)\|_i \leq \zeta_i \|x_1 - x_2\|_1, \quad \forall x_1, \ x_2 \in X_1;$$

(ii) ν_i-Lipschitz continuous in the second argument, if there exists a constant $\nu_i > 0$ such that

$$\|N_i(., y_1) - N_i(., y_2)\|_i \leq \nu_i \|y_1 - y_2\|_2, \quad \forall y_1, \ y_2 \in X_2;$$

(iii) κ_i-strongly accretive in the first argument, if there exists a constant $\kappa_i > 0$ such that

$$\langle N_i(x_1, .) - N_i(x_2, .), J_{q_i}(\eta_i(x_1, x_2))\rangle_i \leq \kappa_i \|x_1 - x_2\|_1^{q_1}, \quad \forall x_1, \ x_2 \in X_1;$$

(iv) ϑ_i-strongly accretive in the second argument, if there exists a constant $\vartheta_i > 0$ such that

$$\langle N_i(., y_1) - N_i(., y_2), J_{q_i}(\eta_i(y_1, y_2))\rangle_i \leq \vartheta_i \|y_1 - y_2\|_2^{q_2}, \quad \forall y_1, \ y_2 \in X_2.$$

Definition 6.3.2. For $i \in \{1, 2\}$, let $P_i : X_1 \times X_2 \to X_i$ be a single-valued mapping and $S_i : X_1 \to \mathrm{CB}(X_1)$, $T_i : X_2 \to \mathrm{CB}(X_2)$ are set-valued mappings. Then P_i is

(i) ϵ_i-Lipschitz continuous in the first argument with respect to S_i, if there exists a constant $\epsilon_i > 0$ such that

$$\|P_i(u_1, .) - P_i(u_2, .)\|_i \leq \epsilon_i \|u_1 - u_2\|_1, \quad \forall (x, y) \in X_1 \times X_2, \ u_1 \in S_1(x), \ u_2 \in S_2(x);$$

(ii) σ_i-Lipschitz continuous in the second argument with respect to T_i, if there exists a constant $\sigma_i > 0$ such that

$$\|P_i(., v_1) - P_i(., v_2)\|_i \leq \sigma_i \|v_1 - v_2\|_2, \quad \forall (x, y) \in X_1 \times X_2, \ v_1 \in T_1(y), \ v_2 \in T_2(y).$$

Now, we will prove the existence of solutions for the system (6.3.1) and the convergence of iterative sequences generated by Algorithm 6.3.1.

Theorem 6.3.1. In problem (6.3.1), suppose that for each $x, y \in X_1 \times X_2$, $F_1(x) \subseteq f_1(X_1)$, $F_2(y) \subseteq f_2(X_2)$, where $F_1 : X_1 \to 2^{X_1}$ and $F_2 : X_2 \to 2^{X_2}$ are the set-valued mappings defined by (6.3.4) and (6.3.5). Suppose that S_1 and S_2 are $\mathcal{D}_1$-Lipschitz continuous with constants l_{S_1} and l_{S_2}, respectively, T_1 and T_2 are $\mathcal{D}_2$-Lipschitz continuous with constants l_{T_1} and l_{T_2}, respectively. For each $i \in \{1,2\}$, assume that

(i) A_i is α_i-expansive, B_i is β_i-Lipschitz continuous and η_i is τ_i-Lipschitz continuous;

(ii) f_i is λ_{f_i}-Lipschitz continuous and ξ_{f_i}-strongly accretive;

(iii) $H_i(A_i, B_i, C_i)$ is r_i-Lipschitz continuous with respect to A_i, s_i-Lipschitz continuous with respect to B_i and t_i-Lipschitz continuous with respect to C_i, respectively;

(iv) N_i is ξ_i-strongly accretive in the i^{th} argument, ζ_i-Lipschitz continuous in the first argument and ν_i-Lipschitz continuous in the second argument;

(v) P_i is ϵ_i-Lipschitz continuous in the first argument with respect to S_i and σ_i- Lipschitz continuous in the second argument with respect to T_i;

In addition

$$\begin{cases} 0 < \frac{\tau_1^{q_1-1}k_1}{\xi_{f_1}}[L_1 + \rho_1\epsilon_1 l_{S_1}] + \frac{\tau_2^{q_2-1}k_2\rho_2}{\xi_{f_2}}[\zeta_2 + \epsilon_2 l_{S_2}] < 1, \\ 0 < \frac{\tau_1^{q_1-1}k_1\rho_1}{\xi_{f_1}}[\nu_1 + \sigma_1 l_{T_1}] + \frac{\tau_2^{q_2-1}k_2}{\xi_{f_2}}[L_2 + \rho_2\sigma_2 l_{T_2}] < 1, \end{cases} \tag{6.3.12}$$

where

$$\begin{aligned} L_1 &= \Big[(r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1} - q_1\rho_1\xi_1 + q_1\rho_1\zeta_1\{(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1} + \tau_1^{q_1-1}\} \\ &\quad + \rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\Big]^{\frac{1}{q_1}}, \\ L_2 &= \Big[(r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2} - q_2\rho_2\xi_2 + q_2\rho_2\nu_2\{(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1} + \tau_2^{q_2-1}\} \\ &\quad + \rho_2^{q_2}C_{q_2}\nu_2^{q_2}\Big]^{\frac{1}{q_2}}, \\ k_1 &= \frac{\tau_1^{q_1-1}}{\mu_1\alpha_1^{q_1} - \gamma_1\beta_1^{q_1} + \delta_1 - \rho_1 m_1}, \ \mu_1\alpha_1^{q_1} - \gamma_1\beta_1^{q_1} + \delta_1 > \rho_1 m_1, \ \mu_1 > \gamma_1, \alpha_1 > \beta_1, \text{ and} \\ k_2 &= \frac{\tau_2^{q_2-1}}{\mu_2\alpha_2^{q_2} - \gamma_2\beta_2^{q_2} + \delta_2 - \rho_2 m_2}, \ \mu_2\alpha_2^{q_2} - \gamma_2\beta_2^{q_2} + \delta_2 > \rho_2 m_2, \ \mu_2 > \gamma_2, \alpha_2 > \beta_2. \end{aligned}$$

Then the system (6.3.1) has a solution $(x, y, u, v, w, z) \in X$, and the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{z_n\}$ generated by Algorithm 6.3.1 converge strongly to x, y, u, v, w and z, respectively.

Proof. Since S_1 and S_2 are $\mathcal{D}_1$-Lipschitz continuous with constants l_{S_1} and l_{S_2}, respectively, T_1 and T_2 are $\mathcal{D}_2$-Lipschitz continuous with constants l_{T_1} and l_{T_2}, respectively. It follows from (6.3.8)-(6.3.11) that

$$\|u_{n+1}-u_n\|_1 \leq \left(1+\frac{1}{n+1}\right) \mathcal{D}_1(S_1(x_{n+1}), S_1(x_n))$$

$$\leq \left(1+\frac{1}{n+1}\right) l_{S_1} \|x_{n+1}-x_n\|_1, \tag{6.3.13}$$

$$\|v_{n+1}-v_n\|_2 \leq \left(1+\frac{1}{n+1}\right) \mathcal{D}_2(T_1(y_{n+1}), T_1(y_n))$$

$$\leq \left(1+\frac{1}{n+1}\right) l_{T_1} \|y_{n+1}-y_n\|_2, \tag{6.3.14}$$

$$\|w_{n+1}-w_n\|_1 \leq \left(1+\frac{1}{n+1}\right) \mathcal{D}_1(S_2(x_{n+1}), S_2(x_n))$$

$$\leq \left(1+\frac{1}{n+1}\right) l_{S_2} \|x_{n+1}-x_n\|_1, \tag{6.3.15}$$

$$\|z_{n+1}-z_n\|_2 \leq \left(1+\frac{1}{n+1}\right) \mathcal{D}_2(T_2(y_{n+1}), T_2(y_n))$$

$$\leq \left(1+\frac{1}{n+1}\right) l_{T_2} \|y_{n+1}-y_n\|_2, \tag{6.3.16}$$

for $n = 0,\ 1,\ 2, \ldots$.

Since, for each $i = 1, 2$, f_i is ξ_{f_i}-strongly accretive and η_i is τ_i Lipschitz continuous, we have

$$\begin{aligned}
&\|f_1(x_{n+1}) - f_1(x_n)\|_1\ \tau_1^{q_1-1}\|x_{n+1}-x_n\|_1^{q_1-1} \\
&\geq\ \|f_1(x_{n+1}) - f_1(x_n)\|_1\ \|\eta_1(x_{n+1}, x_n)\|_1^{q_1-1} \\
&\geq\ \langle f_1(x_{n+1}) - f_1(x_n),\ J_{q_1}(\eta_1(x_{n+1}, x_n))\rangle_1 \\
&\geq\ \xi_{f_1}\ \|x_{n+1}-x_n\|_1^{q_1},
\end{aligned}$$

which implies that

$$\|x_{n+1}-x_n\|_1 \leq \frac{\tau_1^{q_1-1}}{\xi_{f_1}}\|f_1(x_{n+1}) - f_1(x_n)\|_1. \tag{6.3.17}$$

In the light of (6.3.17), we can obtain

$$\|y_{n+1}-y_n\|_2 \leq \frac{\tau_2^{q_2-1}}{\xi_{f_2}}\|f_2(y_{n+1}) - f_2(y_n)\|_2. \tag{6.3.18}$$

For each $i \in \{1,2\}$, it follows that for $(x,y) \in X_1 \times X_2$, the resolvent operators $R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}$ and $R^{H_2(.,.,.)-\eta_2}_{\rho_2,M_2}$ are $k_1 = \frac{\tau_1^{q_1-1}}{\mu_1\alpha_1^{q_1}-\gamma_1\beta_1^{q_1}+\delta_1-\rho_1 m_1}$ and $k_2 = \frac{\tau_2^{q_2-1}}{\mu_2\alpha_2^{q_2}-\gamma_2\beta_2^{q_2}+\delta_2-\rho_2 m_2}$–Lipschitz continuous, respectively. It follows from (6.3.6) and Theorem 6.2.3 that

$$\begin{aligned}
\|f_1(x_{n+1}) - f_1(x_n)\|_1 = \; & \|R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}\ \{H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& -\rho_1 N_1(x_n, y_n) - \rho_1 P_1(u_n, v_n)\}\\
& -[R^{H_1(.,.,.)-\eta_1}_{\rho_1,M_1}\ \{H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\\
& -\rho_1 N_1(x_{n-1}, y_{n-1}) - \rho_1 P_1(u_{n-1}, v_{n-1})\}]\|_1\\
\leq \; & k_1\|H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& -\rho_1 N_1(x_n, y_n) - \rho_1 P_1(u_n, v_n)\\
& -\{H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\\
& -\rho_1 N_1(x_{n-1}, y_{n-1}) - \rho_1 P_1(u_{n-1}, v_{n-1})\}\|_1\\
= \; & k_1\|H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& -H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\\
& -\rho_1(N_1(x_n, y_n) - N_1(x_{n-1}, y_n))\\
& -\rho_1(N_1(x_{n-1}, y_n) - N_1(x_{n-1}, y_{n-1}))\\
& -\rho_1(P_1(u_n, v_n) - P_1(u_{n-1}, v_{n-1}))\|_1\\
\leq \; & k_1\|H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& -H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\\
& -\rho_1(N_1(x_n, y_n) - N_1(x_{n-1}, y_n))\|_1\\
& +k_1\rho_1\|N_1(x_{n-1}, y_n) - N_1(x_{n-1}, y_{n-1})\|_1\\
& +k_1\rho_1\|P_1(u_n, v_n) - P_1(u_{n-1}, v_{n-1})\|_1.
\end{aligned} \tag{6.3.19}$$

Since, for each $i = 1,2$, $H_i(A_i, B_i, C_i)$ is r_i-Lipschitz continuous with respect to A_i, s_i-Lipschitz continuous with respect to B_i, t_i-Lipschitz continuous with respect to C_i, and f_i is λ_{f_i}-Lipschitz continuous, we have

$$\begin{aligned}
& \|H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n))) - H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\|_1\\
& \quad \leq \|H_1(A_1(f_1(x_n)), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& \qquad -H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_n)), C_1(f_1(x_n)))\|_1\\
& \qquad +\|H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_n)), C_1(f_1(x_n)))\\
& \qquad -H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_n)))\|_1\\
& \qquad +\|H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_n)))\\
& \qquad -H_1(A_1(f_1(x_{n-1})), B_1(f_1(x_{n-1})), C_1(f_1(x_{n-1})))\|_1\\
& \quad \leq (r_1 + s_1 + t_1)\|f_1(x_n) - f_1(x_{n-1})\|_1
\end{aligned}$$

$$\leq \ (r_1+s_1+t_1)\lambda_{f_1}\ \|x_n-x_{n-1}\|_1. \tag{6.3.20}$$

In the light of (6.3.20), we can obtain

$$\|H_2(A_2(f_2(y_n)),B_2(f_2(y_n)),C_2(f_2(y_n)))-H_2(A_2(f_2(y_{n-1})),B_2(f_2(y_{n-1})),C_2(f_2(y_{n-1})))\|_2$$
$$\leq \ (r_2+s_2+t_2)\lambda_{f_2}\ \|y_n-y_{n-1}\|_2. \tag{6.3.21}$$

Since, for each $i=1,2$, N_i is ζ_i-Lipschitz continuous in the first argument and ν_i-Lipschitz continuous in the second argument, we have

$$\|N_1(x_n,y_n)-N_1(x_{n-1},y_n)\|_1 \ \leq \ \zeta_1\|x_n-x_{n-1}\|_1, \tag{6.3.22}$$

$$\|N_2(x_n,y_n)-N_2(x_{n-1},y_n)\|_2 \ \leq \ \zeta_2\|x_n-x_{n-1}\|_1, \tag{6.3.23}$$

$$\|N_1(x_n,y_n)-N_1(x_n,y_{n-1})\|_1 \ \leq \ \nu_1\|y_n-y_{n-1}\|_2, \tag{6.3.24}$$

and

$$\|N_2(x_n,y_n)-N_2(x_n,y_{n-1})\|_2 \ \leq \ \nu_2\|y_n-y_{n-1}\|_2. \tag{6.3.25}$$

By using Lemma 1.4.1, (6.3.20), (6.3.21), ξ_1-strongly accretiveness of N_1 in the first argument and τ_1-Lipschitz continuity of η_1, we have

$$\begin{aligned}
&\|H_1(A_1(f_1(x_n)),B_1(f_1(x_n)),C_1(f_1(x_n)))-H_1(A_1(f_1(x_{n-1})),B_1(f_1(x_{n-1})),C_1(f_1(x_{n-1})))\\
&\qquad -\rho_1(N_1(x_n,y_n)-N_1(x_{n-1},y_n))\|_1^{q_1}\\
&\quad\leq \|H_1(A_1(f_1(x_n)),B_1(f_1(x_n)),C_1(f_1(x_n)))\\
&\qquad -H_1(A_1(f_1(x_{n-1})),B_1(f_1(x_{n-1})),C_1(f_1(x_{n-1})))\|_1^{q_1}\\
&\qquad -q_1\rho_1\langle N_1(x_n,y_n)-N_1(x_{n-1},y_n),J_{q_1}(\eta_1(x_n,x_{n-1}))\rangle_1\\
&\qquad -q_1\rho_1\langle N_1(x_n,y_n)-N_1(x_{n-1},y_n),J_{q_1}[H_1(A_1(f_1(x_n)),B_1(f_1(x_n)),C_1(f_1(x_n)))\\
&\qquad -H_1(A_1(f_1(x_{n-1})),B_1(f_1(x_{n-1})),C_1(f_1(x_{n-1})))]-J_{q_1}(\eta_1(x_n,x_{n-1}))\rangle_1\\
&\qquad +\rho_1^{q_1}C_{q_1}\|N_1(x_n,y_n)-N_1(x_{n-1},y_n)\|_1^{q_1}\\
&\quad\leq (r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1}\ \|x_n-x_{n-1}\|_1^{q_1}-q_1\rho_1\xi_1\ \|x_n-x_{n-1}\|_1^{q_1}\\
&\qquad +q_1\rho_1\|N_1(x_n,y_n)-N_1(x_{n-1},y_n)\|_1\big[\|H_1(A_1(f_1(x_n)),B_1(f_1(x_n)),C_1(f_1(x_n)))\\
&\qquad -H_1(A_1(f_1(x_{n-1})),B_1(f_1(x_{n-1})),C_1(f_1(x_{n-1})))\|_1^{q_1-1}+\|\eta_1(x_n,x_{n-1})\|_1^{q_1-1}\big]\\
&\qquad +\rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\|x_n-x_{n-1}\|_1^{q_1}
\end{aligned}$$

$$
\begin{aligned}
&\leq (r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1}\ \|x_n-x_{n-1}\|_1^{q_1} - q_1\rho_1\xi_1\ \|x_n-x_{n-1}\|_1^{q_1}\\
&\quad +q_1\rho_1\zeta_1\ \|x_n-x_{n-1}\|_1\big[(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1}\ \|x_n-x_{n-1}\|_1^{q_1-1}\\
&\quad +\tau_1^{q_1-1}\|x_n-x_{n-1}\|_1^{q_1-1}\big] + \rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\|x_n-x_{n-1}\|_1^{q_1}\\
&= (r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1}\ \|x_n-x_{n-1}\|_1^{q_1} - q_1\rho_1\xi_1\ \|x_n-x_{n-1}\|_1^{q_1}\\
&\quad +q_1\rho_1\zeta_1\big[(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1}+\tau_1^{q_1-1}\big]\|x_n-x_{n-1}\|_1^{q_1}\\
&\quad +\rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\|x_n-x_{n-1}\|_1^{q_1}\\
&= \big[(r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1} - q_1\rho_1\xi_1 + q_1\rho_1\zeta_1\{(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1}+\tau_1^{q_1-1}\}\\
&\quad +\rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\big]\|x_n-x_{n-1}\|_1^{q_1}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|H_1(A_1(f_1(x_n)),B_1(f_1(x_n)),C_1(f_1(x_n))) - H_1(A_1(f_1(x_{n-1})),B_1(f_1(x_{n-1})),C_1(f_1(x_{n-1})))\\
&\qquad -\rho_1(N_1(x_n,y_n)-N_1(x_{n-1},y_n))\|_1\\
&\quad\leq \big[(r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1} - q_1\rho_1\xi_1 + q_1\rho_1\zeta_1\{(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1}+\tau_1^{q_1-1}\}\\
&\qquad +\rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\big]^{\frac{1}{q_1}}\|x_n-x_{n-1}\|_1 = L_1\|x_n-x_{n-1}\|_1, \qquad (6.3.26)
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 = &\big[(r_1+s_1+t_1)^{q_1}\lambda_{f_1}^{q_1} - q_1\rho_1\xi_1 + q_1\rho_1\zeta_1\{(r_1+s_1+t_1)^{q_1-1}\lambda_{f_1}^{q_1-1}+\tau_1^{q_1-1}\}\\
&+\rho_1^{q_1}C_{q_1}\zeta_1^{q_1}\big]^{\frac{1}{q_1}}.
\end{aligned}
$$

Since, for each $i=1,2$, P_i is ϵ_i-Lipschitz continuous in the first argument and σ_i-Lipschitz continuous in the second argument, we have

$$
\begin{aligned}
&\|P_1(u_n,v_n)-P_1(u_{n-1},v_{n-1})\|_1\\
&\qquad\leq\ \|P_1(u_n,v_n)-P_1(u_{n-1},v_n)\|_1 + \|P_1(u_{n-1},v_n)-P_1(u_{n-1},v_{n-1})\|_1\\
&\qquad\leq\ \epsilon_1\|u_n-u_{n-1}\|_1\ +\ \sigma_1\|v_n-v_{n-1}\|_2\\
&\qquad\leq\ \epsilon_1\left(1+\frac{1}{n}\right)l_{S_1}\ \|x_n-x_{n-1}\|_1\ +\ \sigma_1\left(1+\frac{1}{n}\right)l_{T_1}\ \|y_n-y_{n-1}\|_2\\
&\qquad=\ \epsilon_1 l_{S_1}\left(1+\frac{1}{n}\right)\|x_n-x_{n-1}\|_1 + \sigma_1 l_{T_1}\left(1+\frac{1}{n}\right)\|y_n-y_{n-1}\|_2. \qquad (6.3.27)
\end{aligned}
$$

Similarly, we can obtain

$$
\|P_2(w_n,z_n)-P_2(w_{n-1},z_{n-1})\|_2
$$

$$= \epsilon_2 l_{S_2}\left(1+\frac{1}{n}\right)\|x_n - x_{n-1}\|_1 + \sigma_2 l_{T_2}\left(1+\frac{1}{n}\right)\|y_n - y_{n-1}\|_2. \quad (6.3.28)$$

It follows from (6.3.19), (6.3.24), (6.3.26) and (6.3.27) that

$$\|f_1(x_{n+1}) - f_1(x_n)\|_1 \leq k_1\left[L_1 + \rho_1\left(1+\frac{1}{n}\right)\epsilon_1 l_{S_1}\right]\|x_n - x_{n-1}\|_1$$
$$+ k_1\rho_1\left[\nu_1 + \left(1+\frac{1}{n}\right)\sigma_1 l_{T_1}\right]\|y_n - y_{n-1}\|_2. \quad (6.3.29)$$

Using (6.3.17) and (6.3.29), we have

$$\|x_{n+1} - x_n\|_1 \leq \frac{\tau_1^{q_1-1}k_1}{\xi_{f_1}}\left[L_1 + \rho_1\left(1+\frac{1}{n}\right)\epsilon_1 l_{S_1}\right]\|x_n - x_{n-1}\|_1$$
$$+\frac{\tau_1^{q_1-1}k_1\rho_1}{\xi_{f_1}}\left[\nu_1 + \left(1+\frac{1}{n}\right)\sigma_1 l_{T_1}\right]\|y_n - y_{n-1}\|_2. \quad (6.3.30)$$

Let

$$\|x_{n+1} - x_n\|_1 \leq \theta_{1n}\|x_n - x_{n-1}\|_1 + \theta_{2n}\|y_n - y_{n-1}\|_2, \quad (6.3.31)$$

where

$$\theta_{1n} = \frac{\tau_1^{q_1-1}k_1}{\xi_{f_1}}\left[L_1 + \rho_1\left(1+\frac{1}{n}\right)\epsilon_1 l_{S_1}\right],$$

and

$$\theta_{2n} = \frac{\tau_1^{q_1-1}k_1\rho_1}{\xi_{f_1}}\left[\nu_1 + \left(1+\frac{1}{n}\right)\sigma_1 l_{T_1}\right].$$

By (6.3.7) and Theorem 6.2.3, we have

$$\begin{aligned}\|f_2(y_{n+1}) - f_2(y_n)\|_2 = \|R_{\rho_2,M_2}^{H_2(.,.,.)-\eta_2}&\{H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n)))\\ &-\rho_2 N_2(x_n, y_n) - \rho_2 P_2(w_n, z_n)\}\\ &-[R_{\rho_2,M_2}^{H_2(.,.,.)-\eta_2}\{H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1})))\\ &-\rho_2 N_2(x_{n-1}, y_{n-1}) - \rho_2 P_2(w_{n-1}, z_{n-1})\}]\|_2\end{aligned}$$

$$
\begin{aligned}
\leq\; & k_2\|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& -\rho_2 N_2(x_n, y_n) - \rho_2 P_2(w_n, z_n) \\
& -\{H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1}))) \\
& -\rho_2 N_2(x_{n-1}, y_{n-1}) - \rho_2 P_2(w_{n-1}, z_{n-1})\}\|_2 \\
=\; & k_2\|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& -H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1}))) \\
& -\rho_2(N_2(x_n, y_n) - N_2(x_n, y_{n-1})) \\
& -\rho_2(N_2(x_n, y_{n-1}) - N_2(x_{n-1}, y_{n-1})) \\
& -\rho_2(P_2(w_n, z_n) - P_2(w_{n-1}, z_{n-1}))\|_2 \\
\leq\; & k_2\|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& -H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1}))) \\
& -\rho_2(N_2(x_n, y_n) - N_2(x_n, y_{n-1}))\|_2 \\
& +k_2\rho_2\|N_2(x_n, y_{n-1}) - N_2(x_{n-1}, y_{n-1})\|_2 \\
& +k_2\rho_2\|P_2(w_n, z_n) - P_2(w_{n-1}, z_{n-1})\|_2. \qquad (6.3.32)
\end{aligned}
$$

By using Lemma 1.4.1, (6.3.21), (6.3.25), ξ_2-strongly accretiveness of N_2 in the second argument and τ_2-Lipschitz continuity of η_2, we have

$$
\begin{aligned}
& \|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1}))) \\
& \qquad -\rho_2(N_2(x_n, y_n) - N_2(x_n, y_{n-1}))\|_2^{q_2} \\
& \leq \|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& \quad - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1})))\|_2^{q_2} \\
& \quad -q_2\rho_2\langle N_2(x_n, y_n) - N_2(x_n, y_{n-1}), J_{q_2}(\eta_2(y_n, y_{n-1}))\rangle_2 \\
& \quad -q_2\rho_2\langle N_2(x_n, y_n) - N_2(x_n, y_{n-1}), J_{q_2}[H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& \quad -H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1})))] - J_{q_2}(\eta_2(y_n, y_{n-1}))\rangle_2 \\
& \quad +\rho_2^{q_2} C_{q_2}\|N_2(x_n, y_n) - N_2(x_n, y_{n-1})\|_2^{q_2} \\
& \leq (r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2}\ \|y_n - y_{n-1}\|_2^{q_2} - q_2\rho_2\xi_2\ \|y_n - y_{n-1}\|_2^{q_2} \\
& \quad +q_2\rho_2\|N_2(x_n, y_n) - N_2(x_n, y_{n-1})\|_2\big[\|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) \\
& \quad -H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1})))\|_2^{q_2-1} + \|\eta_2(y_n, y_{n-1})\|_2^{q_2-1}\big] \\
& \quad +\rho_2^{q_2} C_{q_2}\nu_2^{q_2}\|y_n - y_{n-1}\|_2^{q_2} \\
& \leq (r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2}\ \|y_n - y_{n-1}\|_2^{q_2} - q_2\rho_2\xi_2\ \|y_n - y_{n-1}\|_2^{q_2} \\
& \quad +q_2\rho_2\nu_2\ \|y_n - y_{n-1}\|_2\big[(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1}\ \|y_n - y_{n-1}\|_2^{q_2-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\tau_2^{q_2-1}\|y_n - y_{n-1}\|_2^{q_2-1}\big] + \rho_2^{q_2}C_{q_2}\nu_2^{q_2}\|y_n - y_{n-1}\|_2^{q_2} \\
&= (r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2}\ \|y_n - y_{n-1}\|_2^{q_2} - q_2\rho_2\xi_2\ \|y_n - y_{n-1}\|_2^{q_2} \\
&+q_2\rho_2\nu_2\big[(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1} + \tau_2^{q_2-1}\big]\|y_n - y_{n-1}\|_2^{q_2} \\
&+\rho_2^{q_2}C_{q_2}\nu_2^{q_2}\|y_n - y_{n-1}\|_2^{q_2} \\
&= \big[(r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2} - q_2\rho_2\xi_2 + q_2\rho_2\nu_2\{(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1} + \tau_2^{q_2-1}\} \\
&+\rho_2^{q_2}C_{q_2}\nu_2^{q_2}\big]\|y_n - y_{n-1}\|_2^{q_2}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|H_2(A_2(f_2(y_n)), B_2(f_2(y_n)), C_2(f_2(y_n))) - H_2(A_2(f_2(y_{n-1})), B_2(f_2(y_{n-1})), C_2(f_2(y_{n-1}))) \\
&\qquad -\rho_2(N_2(x_n, y_n) - N_2(x_n, y_{n-1}))\|_2 \\
&\le \big[(r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2} - q_2\rho_2\xi_2 + q_2\rho_2\nu_2\{(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1} + \tau_2^{q_2-1}\} \\
&\qquad +\rho_2^{q_2}C_{q_2}\nu_2^{q_2}\big]^{\frac{1}{q_2}}\|y_n - y_{n-1}\|_2 = L_2\|y_n - y_{n-1}\|_2, \qquad (6.3.33)
\end{aligned}
$$

where

$$
\begin{aligned}
L_2 = &\big[(r_2+s_2+t_2)^{q_2}\lambda_{f_2}^{q_2} - q_2\rho_2\xi_2 + q_2\rho_2\nu_2\{(r_2+s_2+t_2)^{q_2-1}\lambda_{f_2}^{q_2-1} + \tau_2^{q_2-1}\} \\
&+\rho_2^{q_2}C_{q_2}\nu_2^{q_2}\big]^{\frac{1}{q_2}}.
\end{aligned}
$$

It follows from (6.3.23),(6.3.28),(6.3.32) and (6.3.33) that

$$
\begin{aligned}
\|f_2(y_{n+1}) - f_2(y_n)\|_2 \le\ & k_2\rho_2\left[\zeta_2 + \left(1+\frac{1}{n}\right)\epsilon_2 l_{S_2}\right]\|x_n - x_{n-1}\|_1 \\
&+ k_2\left[L_2 + \rho_2\left(1+\frac{1}{n}\right)\sigma_2 l_{T_2}\right]\|y_n - y_{n-1}\|_2. \qquad (6.3.34)
\end{aligned}
$$

Using (6.3.18) and (6.3.34), we have

$$
\begin{aligned}
\|y_{n+1} - y_n\|_2 \le\ & \frac{\tau_2^{q_2-1}k_2\rho_2}{\xi_{f_2}}\left[\zeta_2 + \left(1+\frac{1}{n}\right)\epsilon_2 l_{S_2}\right]\|x_n - x_{n-1}\|_1 \\
&+ \frac{\tau_2^{q_2-1}k_2}{\xi_{f_2}}\left[L_2 + \rho_2\left(1+\frac{1}{n}\right)\sigma_2 l_{T_2}\right]\|y_n - y_{n-1}\|_2. \qquad (6.3.35)
\end{aligned}
$$

Let

$$
\|y_{n+1} - y_n\|_2 \le \theta_{3n}\|x_n - x_{n-1}\|_1 + \theta_{4n}\|y_n - y_{n-1}\|_2, \qquad (6.3.36)
$$

where

$$
\theta_{3n} = \frac{\tau_2^{q_2-1}k_2\rho_2}{\xi_{f_2}}\left[\zeta_2 + \left(1+\frac{1}{n}\right)\epsilon_2 l_{S_2}\right],
$$

and

$$\theta_{4n} = \frac{\tau_2^{q_2-1}k_2}{\xi_{f_2}}\left[L_2 + \rho_2\left(1+\frac{1}{n}\right)\sigma_2 l_{T_2}\right].$$

Adding (6.3.31) and (6.3.36), we have

$$\begin{aligned}\|x_{n+1}-x_n\|_1 + \|y_{n+1}-y_n\|_2 &\leq (\theta_{1n}+\theta_{3n})\|x_n - x_{n-1}\|_1 + (\theta_{2n}+\theta_{4n})\|y_n - y_{n-1}\|_2\\ &\leq \theta_n\big(\|x_n - x_{n-1}\|_1 + \|y_n - y_{n-1}\|_2\big), \qquad (6.3.37)\end{aligned}$$

where

$$\theta_n = \max\ \{(\theta_{1n}+\theta_{3n}), (\theta_{2n}+\theta_{4n})\}.$$

Let

$$\theta = \max\ \{(\theta_1+\theta_3), (\theta_2+\theta_4)\},$$

where

$$\theta_1 = \frac{\tau_1^{q_1-1}k_1}{\xi_{f_1}}\left[L_1 + \rho_1\epsilon_1 l_{S_1}\right],\ \theta_2 = \frac{\tau_1^{q_1-1}k_1\rho_1}{\xi_{f_1}}\left[\nu_1 + \sigma_1 l_{T_1}\right],$$

and

$$\theta_3 = \frac{\tau_2^{q_2-1}k_2\rho_2}{\xi_{f_2}}\left[\zeta_2 + \epsilon_2 l_{S_2}\right],\ \theta_4 = \frac{\tau_2^{q_2-1}k_2}{\xi_{f_2}}\left[L_2 + \rho_2\sigma_2 l_{T_2}\right].$$

Then $\theta_n \to \theta$, as $n \to \infty$. By (6.3.12), we know that $0 < \theta < 1$ and so (6.3.37) implies that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences. Thus, there exist $x \in X_1$ and $y \in X_2$ such that $x_n \to x$ and $y_n \to y$, as $n \to \infty$.

Now, we prove that $u_n \to u \in S_1(x), v_n \to v \in T_1(y), w_n \to w \in S_2(x)$ and $z_n \to z \in T_2(y)$. From (6.3.13)-(6.3.16), we know that $\{u_n\}, \{v_n\}, \{w_n\}$ and $\{z_n\}$ are Cauchy sequences. Hence, there exist $u \in X_1, v \in X_2, w \in X_1, z \in X_2$ such that $u_n \to u, v_n \to v, w_n \to w, z_n \to z$, as $n \to \infty$.

Furthermore,

$$\begin{aligned}d(u, S_1(x)) &\leq \|u - u_n\|_1 + d(u_n, S_1(x))\\ &\leq \|u - u_n\|_1 + \mathcal{D}_1(S_1(x_n), S_1(x))\\ &\leq \|u - u_n\|_1 + l_{S_1}\|x_n - x\|_1 \to 0,\ \text{as } n \to \infty.\end{aligned}$$

Which implies that $d(u, S_1(x)) = 0$. Since $S_1(x) \in CB(X_1)$, it follows that $u \in S_1(x)$. Similarly, we can show that $v \in T_1(y),\ w \in S_2(x),\ z \in T_2(y)$. By continuity of

$f_1, f_2, H_1, H_2, A_1, A_2, B_1, B_2, C_1, C_2, \eta_1, \eta_2, N_1, N_2, P_1, P_2, R_{\rho_1,M_1}^{H_1(.,.,.)-\eta_1}, R_{\rho_2,M_2}^{H_2(.,.,.)-\eta_2}$ and Algorithm 6.3.1, we know that x, y, u, v, w, z satisfy the following relation:

$$f_1(x) = R_{\rho_1,M_1}^{H_1(.,.,.)-\eta_1}(H_1(A_1(f_1(x)), B_1(f_1(x)), C_1(f_1(x))) - \rho_1 N_1(x,y) - \rho_1 P_1(u,v)),$$

$$f_2(y) = R_{\rho_2,M_2}^{H_2(.,.,.)-\eta_2}(H_2(A_2(f_2(y)), B_2(f_2(y)), C_2(f_2(y))) - \rho_2 N_2(x,y) - \rho_2 P_2(w,z)).$$

By Lemma 6.3.1, (x, y, u, v, w, z) is the solution of system (6.3.1).
This completes the proof.

Bibliography

Bibliography

[1] Abbas, M., Khan, M. A. and Radenovic, S., *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput., 217, 195-202, (2010).

[2] Abbas, M., Nazir, T. and Radenovic, S., *Common fixed points of four maps in partially ordered metric spaces*, Appl. Math. Lett., 24, 1520-1526, (2011).

[3] Adly, S., *Perturbed algorithms and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl., 201(2), 609-630, (1996).

[4] Agarwal, R. P., Huang, N. J. and Tan, M. Y., *Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions*, Appl. Math. Lett., 17, 345-352, (2001).

[5] Agarwal, R. P., El-Gebeily, M. A. and O'Regan, D., *Generalized contractions in partially ordered metric spaces*, Appl. Anal., 87(1), 109-116, (2008).

[6] Ahmad, R. and Yao, J. C., *System of generalized resolvent equations with corresponding system of variational inclusions*, J. Global Opt., 44, 297-309, (2009).

[7] Ahmad, R., Dilshad, M., Wong, M. M., and Yao, J. C., $H(.,.)$*-cocoercive operator and an application for solving generalized variational inclusions*, Abstract and Applied Analysis, 2011:261534, 12 pp., (2011).

[8] Alam, A., Khan, A. R. and Imdad, M., *Some coincidence theorems for generalized nonlinear contractions in ordered metric spaces with applications*, Fixed Point Theory Appl., 2014:216, 14pp., (2014).

[9] Alam, A. and Imdad, M., *Monotone generalized contractions in ordered metric spaces* (submitted).

[10] Alber, Y. I., *Metric and generalized projection operators in Banach spaces: Properties and applications, Theory and Applications of Nonlinear Operators of Monotone*

and Accretive Type, Lecture notes in Pure and Appl. Math., Marcel Dekker, New York, 178, 15-50, (1996).

[11] Alotaibi, A., Kumar, V. and Hussain, N., *Convergence comparison and stability of Jungck-Kirk type algorithms for common fixed point problems*, Fixed Point Theory Appl., 2013:173, 30pp., (2013).

[12] Altun, I. and Simsek, H., *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl., 2010:621469, 17pp., (2010).

[13] Amini-Harandi, A. and Emami, H., *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., 72, 2238-2242, (2010).

[14] Attouch, H., *Variational Convergence for Functions and Operators*, Applied Mathematics Series, Pitman, London, (1984).

[15] Aubin, J. P., *Mathematical Methods of Game Theory and Economics*, North-Holland, Amsterdam, (1982).

[16] Aubin, J. P. and Cellina, A., *Differential Inclusions*, Springer, Berlin, Germany, (1984).

[17] Aubin, J. P. and Ekeland, I., *Applied Nonlinear Analysis*, Wiley-interscience, New York, (1984).

[18] Aubin, J. P. and Frankowska, H., *Set-Valued Analysis*, Birkhäuser, Berlin, (1990).

[19] Aydi, H. and Karapinar, E., *Triple fixed points in ordered metric spaces*, Bull. of Math. Anal. Appl., 4(1), 197-202, (2012).

[20] Baiocchi, C. and Capelo, A., *Variational and Quasi-variational Inequalities: Applications to Free Boundary Problems*, JohnWiley and Sons, New York, (1984).

[21] Banach, S., *Sur les operations dans les ensembles abslraits et leur aplication aur equations integarls*, Fund. Math., 3, 133-181, (1922).

[22] Bensoussan, A., Goursat, M. and Lions, J. L., *Control implusinnel et inequations quasi-variationnelles stationeries*, Compt. Rend. Acad. Sci. Paris, 276, 1279-1284, (1973).

[23] Bensoussan, A. and Lions, J. L., *Applications of Variational Inequalities in Stochastic Control*, North-Holland Publishing Company, Amsterdam, (1982).

[24] Bensoussan, A. and Lions, J. L., *Impulse Control and Quasi-Variational Inequalities*, Gauthiers-Villars, Bordas, Paris, (1984).

[25] Berinde, V. and Borcut, M., *Tripled fixed points theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., 47, 4889-4897, (2011).

[26] Berzig, M. and Samet, B., *An extension of coupled fixed point's concept in higher dimension and applications*, Comput. Math. Appl., 63(8), 1319-1334, (2012).

[27] Brézis, H., Équations et inéquations non Linéares dans les espaces vectoriels en dualité, *Ann. Inst. Fourier,* 18(1), 115-175, (1968).

[28] Brouwer, L .E. J., *Uber Abbildungen Von Mannigfaltigkeiten*, Math. Ann., 71, 97-115, (1912).

[29] Bryant, V. W., *A remark on a fixed point theorem for iterated mappings*, Amer. Math. Monthly, 75, 399-400, (1968).

[30] Bhaskar, T. G. and Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., 65, 1379-1393, (2006).

[31] Ceng, L. C. and Wen, C. F., *Algorithms for general system of generalized resolvent equations with corresponding system of variational inclusions*, J. Appl. Math., 2012:487394, 24pp., (2012).

[32] Ceng, L. C., Husain, N., Latif, A., and Yao, J. C., *Strong convergence for solving general system of variational inequalities and fixed point problems in Banach spaces*, Jour. Ineq. Appl., 2013:334, 44pp., (2013).

[33] Cesari, L., *Alternative methods in nonlinear analysis*, International conference on differential equations, Ed. by H. A. Antosiewicz, Academic Press, 95-149, (1975).

[34] Chandok, S., Khan, M. S., and Rao, K. P. R., *Some coupled common fixed point theorems for a pair of mappings satisfying contractive condition of Rational Type without monotonocity*, Int. J. Math. Anal., 7(9), 433-440, (2013).

[35] Chandok, S. and Tas, K., *An original coupled coincidence point result for a pair of mappings without MMP*, Jour. Ineq. Appl., 2014:61, 12pp., (2014).

[36] Chang, S. S. and Pang, J. S., *The generalized quasi-variational inequality problem*, Math. Oper. Res., 2, 211-222, (1982).

[37] Chang, S. S., Kim, J. K. and Kim, K. H., *On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces*, J. Math. Anal. Appl., 268, 89-108, (2002).

[38] Chatterjee, S. K., *Fixed point theorems*, Computers Reno, Acad., Bulgar. Sci., 25, 727-730, (1972).

[39] Chauhan, S., *Common fixed point theorems in Menger probabilistic quasi metric spaces through weak compatibility*, Int. Math. Forum, 5(18), 851-859, (2010).

[40] Chang, S. S., Cho, Y. J. and Kang, S. M., *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers, Inc., New York, (2001).

[41] Chen, J. Y., Wong, N. C. and Yao, J. C., *Algorithms for generalized cocomplementarity problems in Banach spaces*, Comput. Math. Appl., 43, 49-54, (2002).

[42] Chi, K. P., *On a fixed point theorem for certain class of maps satisfying a contractive condition depended on an another function*, Lobachevskii J. Math., 30(4), 289-291, (2009).

[43] Chi, K. P. and Thuy, H. T., *A fixed point theorem in 2-metric spaces for a class of maps that satisfy a contractive condition dependent on an another function*, Lobachevskii J. Math., 31(4), 338-346, (2010).

[44] Cho, Y. J., Fang, Y. P., and Huang, N. J., *Algorithms for systems of nonlinear variational inequalities*, J. Korean Math. Soc., 41(2), 489-499, (2004).

[45] Choudhury, B. S. and Kundu, A. *A coupled coincidence point result in partially ordered metric spaces for compatible mappings,* Nonlinear Anal., 73, 2524-2531, (2010).

[46] Chugh, R. and Rathi, S. *Weakly compatible maps in probabilistic metric spaces*, J. Indian Math. Soc., 72, 131-140, (2005).

[47] Cioranescu., I., *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, (1990).

[48] Collatz, L., *Functional Analysis and Numerical Analysis*, Academic Press, New York, (1966).

[49] Ćirić, Lj. B., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., 45, 267-273, (1974).

[50] Ćirić, Lj. B., Cakić, M., Rajović, S. and Ume, J. S., *Monotone generalized contractions in partially ordered metric spaces*, Fixed Point Theory Appl., 2008:131294, 11 pp., (2008).

[51] Crank, J., *Free and Moving Boundary Problems*, Clarendon Press, Oxford, (1984).

[52] Cronion, J., *Fixed Points and Topological Degree in Non-linear Analysis*, American Math. Soc., (1964).

[53] Dalal, S., Khan, M. A. and Chauhan, S., *n-tupled coincidence point theorems in partially ordered metric spaces for compatible mappings*, Abstr. Appl. Anal., 2013:614019, 8 pp., (2014).

[54] Demiling, K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, (1985).

[55] Dien, N. H., *Some remarks on variational-like and quasi-variational-like inequalities*, Bull. Austral. Math. Soc., 46, 335-342, (1992).

[56] Ding, X. P. and Wang, Z. B., *System of set-valued mixed quasi-variational-like inclusions involving H-η-monotone operators in Banach spaces*, Appl. Math. Mech.-Engl. Ed., 30(1), 1-12, (2009).

[57] Doric, D., Kadelburg, Z. and Radenovic, S., *Coupled fixed point results for mappings without mixed monotone property*, Appl. Math. Lett., 25, 1803-1808, (2012).

[58] Duvaut, G. and Lions, J. L., *Inequalities in Mechanics and Physics*, Springer- Verlag, Berlin, (1976).

[59] Ekland, I. and Temam, R., *Convex Analysis and Variational Inequalities*, North-Holland, Amsterdam, (1976).

[60] Fang, Y. P. and Huang, N. J., *H-monotone operator and resolvent operator technique for variational inclusions*, Appl. Math. Comput., 145(2), 795-803, (2003).

[61] Fang, Y. P. and Huang, N. J., *H-monotone operators and system of variational inclusions*, Comm. Appl. Nonlinear Anal., 11(1), 93-101, (2004).

[62] Fang, Y. P. and Huang, N. J., *H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Appl. Math. Lett., 17, 647-653, (2004).

[63] Fang, Y. P., Huang, N. J. and Thompson, H. B., *A new system of variational inclusions with (H,η)-monotone operators in Hilbert spaces*, Comp. Math. Appl., 49(2-3), 365-374, (2005).

[64] Feng, H. R. and Ding, X. P., *A new system of generalized nonlinear quasi-variational-like inclusions with A-monotone operators in Banach spaces*, J. Comput. Appl. Math., 225(2), 365-373, (2009).

[65] Fichera, G., *Problemi elastostatici con vincoli unilaterali: II problema di Signorini con ambigue condizioni al contorno (Italian), Atti Accad. Naz. Lincei Mem.*, Cl. Sci. Fis. Mat. Nat. Sez., 17 (8), 91-140, (1963/1964).

[66] Giannessi, F., *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht, (1999).

[67] Glowinski, R., Lions, J. L. and Tremolieres, R., *Numerical Analysis of Variational Inequalities*, North-Holland Publishing Company, Amsterdam, New York, (1981).

[68] Goebel, K. and Reich, S., *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, (1984).

[69] Gopal, D., Imdad, M., and Vetro, C., *Impact of common property (E.A.) on fixed point theorems in fuzzy metric spaces*, Fixed Point Theory Appl., 2011:297360, 3pp., (2011).

[70] Gordji, M. E. and Ramezani M., *N-fixed point theorems in partially ordered metric spaces*, Preprint.

[71] Gorniewicz, L., *Topological Fixed Point Theory of Multivalued Mappings*, Springer-Verlag, Berlin, (2006).

[72] Grabiec, M., Cho, Y. J. and Radu, V., *On Nonsymmetric Topological and Probabilistic Structures*, New York, Nova Science Publishers, Inc., (2006).

[73] Hadzic, O. and Pap, E., *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publisher, Dordrecht, (2001).

[74] Haghi, R. H., Rezapour, Sh. and Shahzad, N., *Some fixed point generalizations are not real generalizations*, Nonlinear Anal., 74, 1799-1803, (2011).

[75] Harjani, J. and Sadarangani, K., *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., 71, 3403-3410, (2009).

[76] Harjani, J. and Sadarangani, K., *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal., 72, 1188-1197, (2010).

[77] Harjani, J., López, B. and Sadarangani, K., *A fixed point theorem for weakly C-contractive mappings in ordered metric spaces*, Comput. Math. Appl., 61, 790-796, (2011).

[78] Harjani, J., Lopez, B. and Sadarangani, K., *A fixed point theorem for weakly C-contractive mappings in ordered metric spaces*, Comput. Math. Appl., 61, 790-796, (2011).

[79] Hassouni, A. and Moudafi, A., *A perturbed algorithm for variational inequalities*, J. Math. Anal. Appl., 185, 706-712, (1994).

[80] He, X. and Zhou, Y., *An iterative method for a system of generalized mixed quasi-variational inclusions with (A,η)-monotone mappings*, Appl. Math. Comput., 215, 4256-4262, (2010).

[81] Huang, N. J. and Deng, C. X., *Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities*, J. Math. Anal. Appl., 256, 345-359, (2001).

[82] Huang, N. J. and Fang, Y. P., *Generalized m-accretive mappings in Banach spaces,* Journal of Sichuan University, 38(4), 591-592, (2001).

[83] Huang, N. J. and Fang, Y. P., *A new class of generalized variational inclusions involving maximal η-monotone mappings*, Publ. Math. Debrecen, 62(1-2), 83-98, (2003).

[84] Huang, N. J., *Nonlinear implicit quasi-variational inclusions involving generalized m-accretive mappings*, Arch. Inequal. Appl., 2(4), 413-425, (2004).

[85] Husain, S., Gupta, S. and Mishra, V. N., *Generalized $H(.,.)$-η-cocoercive operators and generalized set-valued variational-like inclusions*, Jour. Math., 2013:738491, 10pp., (2013).

[86] Husain, S., Gupta, S. and Mishra, V. N., *Graph convergence for the $H(.,.)$-mixed mapping with an application for solving the system of generalized variational inclusions*, Fixed Point Theory Appl., 2013:304, 21pp., (2013).

[87] Husain, S., Sahper, H. and Sharma, A., *Generalized n-tupled common fixed point theorems for contractive rational type condition*, British J. Math. comp. Sci., 4(5), 735-748, (2014).

[88] Hussain, N., Latif, A. and Shah, M. H., *Coupled and tripled coincidence point results without compatibility*, Fixed Point Theory Appl., 2012:77, 9 pp., (2012).

[89] Imdad, M. and Ali, J., *A general fixed point theorem in fuzzy metric spaces via implicit function*, Jour. of Applied Mathematics and Informatics, 26(3-4), 591-603, (2008).

[90] Imdad, M., Ali, J. and Tanveer, M., *Coincidence and common fixed point theorems for nonlinear contractions in Menger PM-spaces*, Chaos Solitons Fractals, 42, 3121-3129, (2009).

[91] Imdad, M., Soliman, A. H., Choudhury, B. S. and Das, P., *On n-tupled coincidence and common fixed points results in metric spaces*, Jour. of Operators, 2013:532867, 9pp., (2014).

[92] Imdad, M., Alam, A. and Soliman, A. H., *Remarks on a recent general even-tupled coincidence theorem*, J. Adv. Math., 9(1), 1787-1805, (2014).

[93] Jachymski, J., *Equivalent conditions for generalized contractions on (ordered) metric spaces*, Nonlinear Anal., 74, 768-774, (2011).

[94] Jungck, G., *Commuting maps and fixed points*, Amer. Math. Monthly, 83, 261-263, (1976).

[95] Jungck, G., *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., 9, 771-779, (1986).

[96] Jungck, G. and Rhoades, B. E., *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., 29(3), 227-238, (1998).

[97] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc., 60, 71-76, (1968).

[98] Karapinar, E., *Quartet fixed point for nonlinear contractions*, 2010; http://arxiv.org/abs/1106.5472.

[99] Karapinar, E., Berinde, V., *Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Banach J. Math. Anal., 6(1), 74-89, (2012).

[100] Karapinar, E. and Luong, N. V., *Quadruple fixed point theorems for nonlinear contractions*, Comput. Math. Appl., 64(6), 1839-1848, (2012).

[101] Karapinar, E., Shantanawi, W. and Mustafa, Z., *Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces*, J. Appl. Math., 2012:951912, 17pp., (2012).

[102] Karamardian, S., *The nonlinear complementarity problem with applications, part 2*, Journal of Optimization Theory and Applications, 4(3), 167-181, (1969).

[103] Kassay, G. and Kolumban, J., *System of multi-valued variational inequalities*, Publ. Math. Debrecen, 56, 185-195, (2000).

[104] Kazmi, K. R., *Mann and Ishikawa type perturbed iterative algorithms for generalized quasi-variational inclusions*, J. Math. Anal. Appl., 209, 572-584, (1997).

[105] Kazmi, K. R. and Bhat, M. I., *Iterative algorithm for a system of nonlinear variational-like inclusions*, Comput. Math. Appl., 48, 1929-1935, (2004).

[106] Kazmi, K. R. and Khan, F. A., *Auxiliary problems and algorithm for a system of generalized variational-like inequality problems*, *J. Appl. Math. Comput.*, 187, 789-796, (2007).

[107] Kazmi, K. R. and Khan, F. A., *Iterative approximation of a unique solution of a system of variational-like inclusions in real q-uniformly smooth Banach spaces*, Nonlinear Analysis, 67(3), 917-929, (2007).

[108] Kazmi, K. R., Bhat, M. I. and Ahmad, N., *An iterative algorithm based on M-proximal mappings for a system of generalized implicit variational inclusions in Banach spaces*, J. comput. Appl. Math., 233(2), 361-371, (2009).

[109] Kazmi, K. R., Khan, F. A. and Shahzad, M., *A system of generalized variational inclusions involving generalized $H(.,.)$-accretive mapping in real q-uniformaly smooth Banach spaces*, Appl. Math. Comput., 217(23), 9679-9688, (2011).

[110] Kazmi, K. R., Ahmad, N. and Shahzad, M., *Convergence and stability of an iterative algorithm for a system of generalized implicit variational-like inclusions in Banach spaces*, Appl. Math. Comput., 218(18), 9208-9219, (2012).

[111] Kent, D. C. and Richardson, G. D., *Ordered probabilistic metric spaces*, J. Aust. Math. Soc. Ser. A, 46(1), 88-99, (1989).

[112] Kikuchi, N. and Oden, J. T., *Contact Problems in Elasticity, A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadalphia, (1988).

[113] Kinderlehrer, D. and Stampacchia, G., *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[114] Klement, E. P., Mesiar, R. and Pap, E., *Triangular norms*, Kluwer Academic Publishers, Trend in Logic 8, Dordrecht, (2000).

[115] Kohli, J. K. and Vashistha, S., *Common fixed point theorems in probabilistic metric spaces*, Acta Math. Hungar., 115, 37-47, (2007).

[116] Konnov, I. V., *Relatively monotone variational inequalities over product sets*, Oper. Res. Lett., 28, 21-26, (2001).

[117] Kreyszig, E., *Introductory Functional Analysis with Applications*, John Wiley and Sons, New York, (1978).

[118] Kunzi, H. P. A., *Nonsymmetric distances and their associated topologies: about the origin of basic ideas in the area of asymmetric topology*, in Handbook of History of General Topology, Kluwer Academic Publishers, Dordrecht, The Netherlands, 3, 853-968, (2001).

[119] Lakshmikantham, V. and Ciric, L., *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., 70, 4341-4349, (2009).

[120] Lan, H. Y., Cho, Y. J. and Verma, R. U., *Nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings in banach spaces*, Comput. Math. Appl., 51, 1529-1538, (2006).

[121] Lan, H. Y., Kim, J. H. and Cho, Y. J., *On a new system of nonlinear A-monotone multivalued variational inclusions*, J. Math. Anal. Appl., 327, 481-493, (2007).

[122] Lions, J. L. and Stampacchia, G., Variational inequalities, *Comm. Pure Appl. Math.*, 20, 493-519, (1967).

[123] Leggett, R. W. and Williams, L. R., *A fixed point theorem with applications to an infectious disease model*, J. Math. Anal. Appl., 76, 91-97, (1980).

[124] Lu, Z. X., *Common fixed points for converse commuting self maps on a metric space*, Acta Analysis Functionalis Applicata, 4(3), 226-228, (2002).

[125] Luong, N. V., Thuan, N. X. and Hai, T. H., *Coupled fixed point theorems in partially ordered metric spaces depended on another function*, Bull. Math. Anal. Appl., 3, 129-140, (2011).

[126] Luong, N. V. and Thuan, N. X., *Coupled fixed point theorems in partially ordered metric spaces and application*, Nonlinear Anal., 74, 983-992, (2011).

[127] Martin, R. H., *Nonlinear operators and differential equations in Banach spaces*, John Wiley and Sons, New York, (1976).

[128] Menger, K., *Statistical metrics*, Proceedings of the National Academy of Sciences of the United States of America, 28, 535-537, (1942).

[129] Mihet, D., *On the existence and uniqueness of fixed points of Sehgal contractions*, Fuzzy Sets and Systems, 156, 135-141, (2005).

[130] Mihet, D., *A counter example to common fixed point theorem in Menger probabilistic quasi-metric spaces*, J. Nonlinear Sci. Appl., 1(2), 121-122, (2008).

[131] Mihet, D., *A note on a fixed point theorem in Menger probabilistic quasi-metric spaces*, Chaos Solitans Fractals, 40, 2349-2352, (2009).

[132] Mishra, S. N., *Common fixed points of compatible mappings in PM-spaces*, Mathematica Japonica, 36(2), 283-289, (1991).

[133] Mohamad, A., *Probabilistic quasi-metric spaces*, Conference in Topology and Theoretical Computer Science in honour of Peter Collins and Mike Reed, Mathematical Institute, University of Oxford, Oxford, U.K., 7-10, (2006).

[134] Moore, R. E., *Computational Functional Analysis*, Ellis Harwood Limited, New York, (1985).

[135] Moradi, S. and Omid, M., *A fixed point theorem for integral type inequality depending on another function*, Int. J. Math. Anal. (Ruse), 4(29-32), 1491-1499, (2010).

[136] Mosco, U., *Implicit variational problems and quasi-variational inequalities*, In: Lecture Notes in Mathematics, Springer-Verlag, Berlin, 543, 83-156, (1976).

[137] Mustafa, Z., Aydi, H. and Karapinar, E., *Mixed g-monotone property and quadruple fixed point theorems in partial ordered metric space*, Fixed Point Theory Appl., 2012:71, 19pp., (2012).

[138] Nadler, S. B., *Multivalued contraction mapping*, Pacific Journal of Mathematics, 30(3), 457-488, (1969).

[139] Nashine, H. K. and Samet, B., *Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal., 74, 2201-2209, (2011).

[140] Nashine, H. K. and Shatanawi, W., *Coupled common fixed point theorems for a pair of commutating mapping in partially ordered metric spaces*, Comput. Math. Appl., 62, 1984-1993, (2011).

[141] Nashine, H. K. and Altun, I., *A common fixed point theorem on ordered metric spaces*, Bull. Iranian Math. Soc., 38(4), 925-934, (2012).

[142] Něcas, J., Jarusek, J. and Haslinger, J., *On the solution of the variational inequality related to the Signorini problem with small friction*, Boll. Un. Math. Ital., 17, 796-811, (1980).

[143] Nieto, J. J. and Lopez, R. R., *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22, 223-239, (2005).

[144] Nieto, J. J. and Lopez, R. R., *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica (Engl. Ser.), 23(12), 2205-2212, (2007).

[145] Noor, M. A., *On Nonlinear Variational Inequalities, Ph.D. Thesis*, Burnel University, U.K., (1975).

[146] Noor, M. A., *Variational inequalities for a class of contact problems with friction in elastostatistic, Compt. Rend. Math. Rep. Acad. Sci. Canada.*, 5, 127-132, (1983).

[147] Noor, M. A., *General variational inequalities*, Appl. Math. Lett., 1, 119-122, (1988).

[148] Noor, M. A., *Quasi-variational inequalities*, Appl. Math. Lett., 1(4), 367-370, (1988).

[149] Noor, M. A. and Noor, K. I., *Multivalued variational inequalities and resolvent equations*, Math. Comput. Model., 26(7), 109-121, (1997).

[150] Noor, M. A., *Generalized set-valued variational inclusions and resolvent equations*, J. Math. Anal. Appl., 228, 206-220, (1998).

[151] Noor, M. A., Noor, K. I. and Rassias, T. M., *Set-valued resolvent equations and mixed variational inequalities*, J. Math. Anal. Appl., 220, 741-759, (1998).

[152] Noor, M. A., *Generalized quasi-variational inequalities and implicit Wiener- Hopf equation*, Optimization, 45, 197-222, (1999).

[153] Oden, J. T. and Pires, E. B., *Contact problems in elastostatistics with nonlocal friction laws*, TICOM Report, 12, Univ. of Texas, Austin, (1981).

[154] O'Regan, D. and Petrusel, A., *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl., 314(2), 1241-1252, (2008).

[155] Pang, J. S., *Asymmetrical variational inequalities over product of sets: Applications and iterative methods*, Math. Programming, 31, 206-219, (1985).

[156] Pant, B. D. and Chauhan, S., *Fixed point theorems in Menger probabilistic quasi metric spaces using weak compatibility*, Internat. Math. Forum, 5(6), 283-290, (2010).

[157] Parida, J., Sahoo, M. and Kumar, A., *A variational-like inequality problem*, Bull. Austral. Math. Soc., 39, 225-231, (1989).

[158] Peng, J. and Zhu, D., *A new system of generalized mixed-quasi-variational inclusions with (H,η)-monotone operators*, J. Math. Anal. Appl., 327, 175-187, (2007).

[159] Peng, J. W. and Zhu, D. L., *A system of variational inclusions with p-η- accretive operators*, J. Comput. Appl. Math., 216, 198-209, (2008).

[160] Petershyn, W. V., *A characterization of strict convexity of Banach spaces and other uses of duality mappings*, J. Funct. Anal., 6, 282-291, (1970).

[161] Popa, V., *A fixed point theorem for mapping in d-complete topological spaces*, Mathematica Moravica, 3, 43-48, (1999).

[162] Radenovic, S. and Kadelburg, Z., *Generalized weak contractions in partially ordered metric spaces*, Comput. Math. Appl., 60, 1776-1783, (2010).

[163] Radenovic, S., *A note on tripled coincidence and tripled common fixed point theorems in partially ordered metric spaces*, Appl. Math. Comput., 236, 367-372, (2014).

[164] Radu, V., *Lectures on Probabilistic Aanalysis, Surveys, Lecture Notes and Monographs*, Series on Probability, Statistics and Applied Mathematics, Universitatea din Timisoara, (1994).

[165] Rakotch, E., *A note on contractive mappings*, Proc. Amer. Math. Soc., 13, 459-465, (1962).

[166] Ran, A. C. M. and Reurings, M. C. B., *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132, 1435-1443, (2004).

[167] Rashwan, R. A. and Hedar, A., *On common fixed point theorems of compatible maps in Menger spaces*, Demonst. Math., 31, 537-546, (1998).

[168] Rashwan, R. A. and Saleh, S. M., *Quadruple fixed point theorems for compatible type mappings without mixed g-monotone property in partially ordered b-metric spaces*, Math. Sci., DOI 10.1007/s40096-014-0134-5, (2014).

[169] Rasouli, S. H. and Malekshah, M. H., *Coupled fixed point results for mappings without mixed monotone property in partially ordered G-metric spaces*, Jour. Egypt. Math. Soc., 22, 471-475, (2014).

[170] Razani, A. and Shirdaryazdi, M., *A common fixed point theorem of compatible maps in Menger space*, Chaos, Solitions Fractals, 32, 26-34, (2007).

[171] Rezaiyan, R., Cho, Y. J. and Saadati, R., *A common fixed point theorem in Menger probabilistic quasi-metric spaces*, Chaos Solitans and Fractals, 37, 1153-1157, (2008).

[172] Riesz, F., Sur une espáce de géométric analytique des systèmes des fonctions sommables, *Comptes Rendus de l̀Acadeémie des Sciences, (Paris),* 144, 1409-1411, (1997).

[173] Roldán, A., Martínez-Moreno, J. and Roldán, C., *Multidimensional fixed point theorems in partially ordered metric spaces*, Jour. of Math. Anal. and Appl., 396(2), 536-545, (2012).

[174] Samet, B. and Vetro, C., *Coupled fixed point, F-invariant set and fixed point of N-order,* Ann. Funct. Anal., 1(2), 46-56, (2010).

[175] Samet, B., *Coupled fixed point theorems for a generalized Meir-Keeler contractions in partially ordered metric spaces*, Nonlinear Anal., 72, 4508-4517, (2010).

[176] Sastry, K. P. R., Naidu, G. A., Prasad, P. V. S. and Sastry, S. S. A., *A fixed point theorem in Menger PQM-spaces using weak compatibility*, Intern. Math. Forum, 5(52), 2563-2568, (2010).

[177] Schauder, J., *Der fixpunksatz functional raumen*, Studia Math., 2, 171-180, (1930).

[178] Schweizer, B., and Sklar, A., *Statistical Metric Spaces*, Pacific Jour. Math., 10, 313-334, (1960).

[179] Schweizer, B., and Sklar, A., *Probabilistic Metric Spaces*, Elsevier, North Holland, New York, (1983).

[180] Sedghi, S., Dosenovic, T. Z. and Shobhe, N., *Common fixed point theorems in Menger probabilistic quasi metric spaces*, Fixed Point Theory Appl., 2009:546273, 11pp., (2009).

[181] Sehgal, V. M. and Bharucha-Reid, A. T., *Fixed point of contraction mappings on probabilistic metric spaces*, Math. Syst. Theory, 6, 97-102, (1972).

[182] Sessa, S., *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.), 32(46), 149-153, (1982).

[183] Siddiqi, A. H. and Ansari, Q. H., *An iterative mdethod for generalized variational inequalities*, Math. Japonica, 34, 475-481, (1989).

[184] Siddiqi, A. H. and Ansari, Q. H., *Strongly nonlinear quasi-variational inequalities*, J. Math. Anal. Appl., 149, 440-450, (1990).

[185] Siddiqi, A. H., Ansari, Q. H. and Ahmad, R., *On generalized variational-like inequalities*, Indian J. Pure Appl. Math., 26(12), 1135-1141, (1995).

[186] Singh, S. L. and Pant, B. D., *Common fixed points of weakly commuting mappings on Non-Archimedean Menger spaces*, Vikram Math. J., 6, 27-31, (1986).

[187] Singh, B. and Jain, S., *A fixed point theorem in Menger space through weak compatibility*, Jour. Math. Anal. Appl., 301, 439-448, (2005).

[188] Singh, B. and Jain, S., *A fixed point theorem in Menger spaces through weak compatibility*, J. Math. Anal. Appl., 301, 439-448, (2005).

[189] Singh, B. and Jain, S., *Semicompatibility and fixed point theorems in fuzzy metric space via implicit relation*, Internat. Jour. of Math. and Mathematical sci., 16, 2617-2629, (2005).

[190] Smart, D. R., Fixed Point Theorems, Cambridge University Press, (1974).

[191] Soliman, A. H., Imdad, M. and Alam, A., *Even-tupled coincidence theorems in generalized complete metric spaces,* Electron. J. Math. Anal. Appl. (accepted).

[192] Stampacchia, G., *Formes bilineaires coercivities sur les ensembles convexes,* Compt. Rend. Acad. Sci. Paris, 258, 4413-4416, (1964).

[193] Tseng, P., *Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming,* Mathematical Programming, 48(2), 249-263, (1990).

[194] Turinci, M., *Ran-Reurings fixed point results in ordered metric spaces,* Libertas Math., 31, 49-55, (2011).

[195] Tychonoff, A., *Ein fixpunktsatz,* Math. Ann., 111, 767-776, (1935).

[196] Verma, R. U., *Projection methods, algorithms, and a new system of nonlinear variational inequalities,* Comput. Math. Appl., 41, 1025-1031, (2001).

[197] Verma, R. U., *Generalized system for relaxed cocoercive variational inequalities and projection methods,* J. Optim. Theory Appl., 121, 203-210, (2004).

[198] Verma, R. U., *A-monotonicity and applications to nonlinear variational inclusion problems,* J. Appl. Math. Stoch. Anal., 17(2), 193-195, (2004).

[199] Verma, R. U., *General system of A-monotone nonlinear variational inclusion problems with Applications,* J. Optim. theory Appl., 131(1), 151-157, (2006).

[200] Verma, R. U., *Sensitivity analysis for generalized strongly monotone variational inclusions based on the* (A, η)*-resolvent operator technique,* Appl. Math. Lett., 19(12), 1409-1413, (2006).

[201] Verma, R. U., *General system of* (A, η)*-monotone variational inclusion problems based on generalized hybrid iterative algorithm,* Nonlinear Analysis: Hybrid Systems, 1(3), 326-335, (2007).

[202] Wang, Z. B. and Ding X. P., $(H(.,.)$-$\eta)$*-accretive operators with an application for solving set-valued variational inclusions in Banach spaces,* Comp. Math. Appl., 59, 1559-1567, (2010).

[203] Xia, F. Q. and Huang, N. J., *Variational inclusions with a general* H*-monotone operator in Banach spaces,* Comput. Math. Appl., 54(1), 24-30, (2007).

[204] Xu, H.K., Inequalities in Banach spaces with applications, Nonlinear Anal., 16, 1127-1138, (1991).

[205] Xu, Z. and Roach, G. F., *Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces*, J. Math. Anal. Appl., 157, 189- 210, (1991).

[206] Yan, W. Y., Fang, Y. P. and Huang, N. J., *A new system of set-valued variational inclusions with (H, η)-monotone operators in Hilbert spaces*, Math. Inequal. Appl., 8(3), 537-546, (2005).

[207] Zeidler, E., *Nonlinear Functional Analysis and its Applications, II/A Nonlinear Operators*, Springer-Verlag, New York, (1986).

[208] Zhang, Q. B., *Generalized implicit variational-like inclusion problems involving G-η-monotone mappings*, Appl. Math. Lett., 20(2), 216-221, (2007).

[209] Zou, Y. Z. and Huang, N. J., *$H(.,.)$-accretive operator with an application for solving variational inclusions in Banach spaces*, Appl. Math. Comput., 204(2), 809-816, (2008).

[210] Zou, Y. Z., and Huang, N. J., *A new system of variational inclusions involving $H(.,.)$-accretive operator in Banach spaces*, Appl. Math. Comput., 212, 135-144, (2009).

List of Publications

Available online at http://scik.org
Advances in Fixed Point Theory, 3 (2013), No. 3, 476-492
ISSN: 1927-6303

GENERALIZED n-TUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES INVOLVING AN ICS MAP

SHAMSHAD HUSAIN* AND HUMA SAHPER

Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India

Abstract. In this paper, we prove results on n-tupled fixed point (for even n) for mapping having the mixed monotone property in partially ordered metric spaces involving an ICS map. These results are the generalizations of the main results of Luong et al. (Bull. Math. Anal. Appl. 3, 129-140, 2011).

Keywords: Partially ordered set; metric space; mixed monotone property; ICS map; n-tupled fixed point.

2000 AMS Subject Classification: 54H10; 54H25

1. Introduction

Fixed point theory is one of the famous and traditional theories in mathematics and has large number of applications. The Banach contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Generalization of the above principle has been a heavily investigated branch of mathematics. Existence of fixed points in partially ordered metric spaces was investigated in 2004 by

*Corresponding author

Received May 11, 2013

Hindawi Publishing Corporation
Journal of Operators
Volume 2013, Article ID 391474, 5 pages
http://dx.doi.org/10.1155/2013/391474

Research Article

Common Fixed Point Theorems for Conversely Commuting Mappings Using Implicit Relations

Sunny Chauhan[1] and Huma Sahper[2]

[1] *Near Nehru Training Centre, H. No. 274, Nai Basti B-14, Bijnor, Uttar Pradesh 246701, India*
[2] *Department of Applied Mathematics, Z. H. College of Engineering and Technology, Aligarh Muslim University, Aligarh, Uttar Pradesh 202002, India*

Correspondence should be addressed to Sunny Chauhan; sun.gkv@gmail.com

Received 24 August 2013; Accepted 25 October 2013

Academic Editor: Haiyan Wang

Copyright © 2013 S. Chauhan and H. Sahper. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The object of this paper is to utilize the notion of conversely commuting mappings due to Lü (2002) and prove some common fixed point theorems in Menger spaces via implicit relations. We give some examples which demonstrate the validity of the hypotheses and degree of generality of our main results.

1. Introduction

In 1986, Jungck [1] introduced the notion of compatible mappings in metric space. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides continuity of at least one of the mappings. Later on, Jungck and Rhoades [2] studied the notion of weakly compatible mappings and utilized it as a tool to improve commutativity conditions in common fixed point theorems. Many mathematicians proved several fixed point results in Menger spaces (see, e.g., [3–9]). In 2002, Lü [10] presented the concept of the converse commuting mappings as a reverse process of weakly compatible mappings and proved common fixed point theorems for single-valued mappings in metric spaces (also see [11]). Recently, Pathak and Verma [12, 13], Chugh et al. [14], and Chauhan et al. [15] proved some interesting common fixed point theorems for converse commuting mappings.

In this paper, we prove some unique common fixed point theorems for two pairs of converse commuting mappings in Menger spaces by using implicit relations.

2. Preliminaries

Definition 1 (see [16]). A t-norm is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying

(T1) $\Delta(a,1) = a$, $\Delta(0,0) = 0$;

(T2) $\Delta(a,b) = \Delta(b,a)$;

(T3) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a$, $d \geq b$;

(T4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for all a, b, c in $[0,1]$.

Examples of t-norms are $\Delta(a,b) = \min\{a,b\}$, $\Delta(a,b) = ab$, and $\Delta(a,b) = \max\{a+b-1,0\}$.

Definition 2 (see [16]). A real valued function f on the set of real numbers is called a distribution function if it is nondecreasing, left continuous with $\inf_{u\in\mathbb{R}} f(u) = 0$ and $\sup_{u\in\mathbb{R}} f(u) = 1$.

We shall denote by $\mathfrak{F}$ the set of all distribution functions defined on $(-\infty,\infty)$, while $H(t)$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases} \tag{1}$$

If X is a nonempty set, $\mathscr{F} : X \times X \to \mathfrak{F}$ is called a probabilistic distance on X and the value of $\mathscr{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 3 (see [17]). A probabilistic metric space is an ordered pair $(X,\mathscr{F})$, where X is a nonempty set of elements

Hindawi Publishing Corporation
Chinese Journal of Mathematics
Volume 2014, Article ID 957482, 7 pages
http://dx.doi.org/10.1155/2014/957482

Research Article

Algorithm for Solving a New System of Generalized Nonlinear Quasi-Variational-Like Inclusions in Hilbert Spaces

Shamshad Husain, Sanjeev Gupta, and Huma Sahper

Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh 202002, India

Correspondence should be addressed to Sanjeev Gupta; guptasanmp@gmail.com

Received 10 September 2013; Accepted 5 November 2013; Published 5 February 2014

Academic Editors: Q. Guo and X.-G. Li

Copyright © 2014 Shamshad Husain et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce and study a new system of generalized nonlinear quasi-variational-like inclusions with $H(\cdot,\cdot)$-cocoercive operator in Hilbert spaces. We suggest and analyze a class of iterative algorithms for solving the system of generalized nonlinear quasi-variational-like inclusions. An existence theorem of solutions for the system of generalized nonlinear quasi-variational-like inclusions is proved under suitable assumptions which show that the approximate solutions obtained by proposed algorithms converge to the exact solutions.

1. Introduction

Variational inclusion problems are important generalization of classical variational inequalities and have wide applications to many fields including mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences; see, for example, [1]. For these reasons, various variational inclusions have been intensively studied in recent years. Many efficient ways have been studied to find solutions for variational inclusions. Those methods include the projection method and its various forms, linear approximation, descent and Newton's method, and the method based on auxiliary principle technique. The method based on the resolvent operator technique is a generalization of the projection method and has been widely used to solve variational inclusions. For details, we refer to see [2–19].

Recently, Fang and Huang, Kazmi and Khan, and Lan et al. investigated several resolvent operators for generalized operators such as H-monotone [3, 17], H-accretive [4], A-maximal relaxed accretive [14], (H,η)-monotone [5], (A,η)-accretive [13], (P,η)-proximal point [8], and (P,η)-accretive [9] operators. Very recently, Zou and Huang [19] introduced and studied $H(\cdot,\cdot)$-accretive operators, Kazmi et al. [10–12] introduced and studied generalized $H(\cdot,\cdot)$-accretive operators and $H(\cdot,\cdot)$-η-proximal point mapping, and Xu and Wang [18] introduced and studied $(H(\cdot,\cdot),\eta)$-monotone operators. Ahmad et al. [2, 8] introduced and studied $H(\cdot,\cdot)$-cocoercive operators, showed some properties of the resolvent operator for the $H(\cdot,\cdot)$-cocoercive operators, and obtained an application for solving variational inclusions in Hilbert spaces. They also gave some examples to illustrate their results.

Inspired and motivated by the researches going on in this area, we introduce and discuss a new system of generalized nonlinear quasi-variational-like inclusions involving $H(\cdot,\cdot)$-cocoercive operators in Hilbert spaces. By using the resolvent operators associated with $H(\cdot,\cdot)$-cocoercive operators due to Ahmad et al. [2], we prove that the approximate solutions obtained by the iterative algorithms converge to the exact solutions of our system of generalized nonlinear quasi-variational-like inclusions. Our results can be viewed as an extension and generalization of some known results in the literature.

2. Preliminaries

Throughout this paper, we suppose that X is a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$, 2^X (resp., $CB(X)$) is

Journal Nonlinear Analysis and Application 2014 (2014) 1-9

ISPACS
International Scientific Publications and Consulting Services

Available online at www.ispacs.com/jnaa
Volume 2014, Year 2014 Article ID jnaa-00142, 9 Pages
doi:10.5899/2014/jnaa-00142
Research Article

JNAA
Journal of Nonlinear Analysis and Application

A common fixed point theorem for weakly compatible mappings in Menger probabilistic quasi metric space

Badridatt Pant[1], Sunny Chauhan[2] *, Suneel Kumar[3], Huma Sahper[4]

(1) *Government Degree College, Champawat-262523, Uttarakhand, India.*

(2) *Near Nehru Training Centre, H. No. 274, Nai Basti B-14, Bijnor-246701, Uttar Pradesh, India.*

(3) *R. H. Government Postgraduate College, Kashipur-244713, U. S. Nagar, Uttarakhand, India.*

(4) *Department of Applied Mathematics, Z. H. College of Engineering and Techonology, A. M. U. Aligarh-202002, Uttar Pradesh, India.*

Copyright 2014 © Badridatt Pant, Sunny Chauhan, Suneel Kumar and Huma Sahper. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we prove a common fixed point theorem for finite number of self mappings in Menger probabilistic quasi metric space. Our result improves and extends the results of Rezaiyan et al. [A common fixed point theorem in Menger probabilistic quasi-metric spaces, Chaos, Solitons and Fractals 37 (2008) 1153-1157.], Miheţ [A note on a fixed point theorem in Menger probabilistic quasi-metric spaces, Chaos, Solitons and Fractals 40 (2009) 2349-2352], Pant and Chauhan [Fixed points theorems in Menger probabilistic quasi metric spaces using weak compatibility, Internat. Math. Forum 5 (6) (2010) 283-290] and Sastry et al. [A fixed point theorem in Menger PQM-spaces using weak compatibility, Internat. Math. Forum 5 (52) (2010) 2563-2568].

Keywords: t-norm, Menger probabilistic quasi metric space, Weakly compatible mappings, Fixed point.

1 Introduction

Classical metrical fixed point theory plays an important role in general topology. In 1942, Menger [10] introduced the notion of probabilistic metric spaces as a generalization of metric spaces. Since then, the theory of probabilistic metric space has developed in many directions [1, 20]. In 1989, Kent and Richardson [7] introduced the class of probabilistic quasi-metric spaces (briefly, PQM-spaces) and proved common fixed point theorems. The study of fixed points of mappings in probabilistic quasi metric spaces is in nascent stage.

With a view to improve commutativity conditions in fixed point theorems, Sessa [22] introduced the notion of weakly commuting mappings. Inspired by this concept, Jungck [5] weakened the notion of weak commutativity by introducing compatible mappings. Further, Jungck and Rhoades [6] introduced the notion of weak compatibility which is the most general among all commutativity concepts. Many authors formulated the definitions of weakly commuting [25], compatibility [14] and weakly compatible mappings [24] in framework of probabilistic settings and proved several fixed point results. Fixed point theorems in PQM-spaces have appeared in [2, 12, 13, 15, 17, 18, 19, 21, 23]. The theory of quasi metric spaces can be used as an efficient tool to solve so many several problems like theoretical computer science, approximation theory and topological algebra (see, for instance [3, 9, 15]).

*Corresponding author. Email address: sun.gkv@gmail.com, Tel:+91-9412452300

British Journal of Mathematics & Computer Science
4(5): 735-748, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Generalized n-Tupled Common Fixed Point Theorems for Contractive Rational Type Condition

Shamshad Husain*[1] Huma Sahper[1] and Anupam Sharma[2]

[1] *Department of Applied Mathematics, Faculty of Engg. and Tech., Aligarh Muslim University, Aligarh, India*
[2] *Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh, India*

Original Research Article

Received: 02 July 2013
Accepted: 23 October 2013
Published: 09 December 2013

Abstract

Aims/ objectives: In this paper, we prove results on n-tupled coincidence point (for even n) for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type in partially ordered metric spaces. Our main theorem improves the corresponding results of Chandok *et al.* (Int. Jour. of Math. Anal., Vol. 7, 2013, No. 9, 433-440).

Keywords: Partially ordered set; compatible mapping; mixed monotone property; n-tupled coincidence point; n-tupled fixed point.

2010 Mathematics Subject Classification: 46T99, 47H10, 54H25.

1 Introduction

In recent years, an extension of Banach's contraction principle has been considered by many authors in different metric spaces. It has fruitful applications within as well as outside mathematics. Generalizations of this principle continues to be an active area of research. Many authors have extended this theorem employing relatively more general contractive conditions ensuring the existence of a fixed point. The investigation of fixed points in ordered metric spaces is a relatively new development which appears to have its origin (in 2004) in the paper of Ran and Reurings [1] which was well complimented by the paper of Nieto and Lopez [2]. For similar other results in ordered metric spaces, one can be referred to ([1]-[23]).

In [3], Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ wherein $(X, \preceq, d)$ be a partial metric space and also proved some coupled fixed point theorems in partially ordered complete metric spaces. In 2009, Bhaskar and Ćirić [4] proved coupled coincidence and coupled fixed point theorems for nonlinear contractive mappings in

*Corresponding author: E-mail: s_husain68@yahoo.com